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TECHNICAL NOTE

DELTA-PERTURBATION EXPANSION FOR THE NORMAL FLOW DEPTH PROBLEM IN RECTANGULAR CHANNELS

AMARA L.[1], ACHOUR B. [2]

[1] Associate Professor, Department of Civil Engineering and Hydraulics, LGCE Laboratory, University of Jijel, Ouled Aissa, Jijel, Algeria.
[2] Professor, Research Laboratory in Subterranean and Surface Hydraulics (LARHYSS), University of Biskra, Algeria.

(*) lyes.amara@univ-jijel.dz

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ABSTRACT

The normal flow depth is an important parameter in the design and analysis of free-surface flow problems for both man-made and natural channels. The implicit character of the governing Manning's equation in rectangular-shaped channels requires a trial and error process for the computation of the normal flow depth, which sometimes leads to a large sequence computation. In this paper, a novel exact analytical approach based on the δ-perturbation expansion is proposed as a solution to the issue of computing normal depth in channels with rectangular cross sections. For practical purposes, the explicit analytical expression obtained provides a particularly precise solution. To show the application of the suggested approach, some examples are presented as illustrative examples.

Keywords: Normal flow depth, Rectangular channel, Analytical solution, Delta-perturbation method.

INTRODUCTION

In engineering problems related to free-surface flow in open channels, the accurate computation of the normal flow depth is of capital importance and even plays a crucial role. Its determination for different-shaped channel profiles is based on flow resistance formulas, and for that, the use of Manning's equation is widespread in practice. However, even for basic cross-section profiles, such as rectangular channels, the computation of the normal depth forms an obstacle due to the implicit character of Manning's formula. For that, the use of graphical and numerical procedures is often needed.

For an explicit computation of the normal flow depth in rectangular channels, several proposals have been made in the past. Along with explicit approximate solutions based on correlation and curve-fitting analysis (Barr and Das, 1986; Srivastava, 2006; Vatankhah, 2016), some attempts have focused on an analytical solution for the inversion of the nonlinear implicit Manning's equation. Swamee and Rathie (2004) approached the problem using Lagrange's inversion theorem. The explicit solution was then presented in the form of an infinite series separately for the cases of a wide and narrow rectangular channel. Later, Ferro and Sciacca (2017) used the same theorem to derive an infinite series expansion, but for the wide rectangular channel case only. It was shown that the series expansion obtained by Lagrange's inversion theorem converges only for a ratio of the normal depth to the channel width $\eta < 1.807$. From a theoretical viewpoint, the divergence of Lagrange's series expansion restricts its use and limits its generalization. It therefore seems necessary to adopt a more general analytical approach. Recently, the so-called δ-perturbation method was successfully applied by Amara and Achour (2023) to the critical flow depth problem in trapezoidal channels.

In the present paper, a new analytical approach based on the δ-perturbation expansion series is applied to the normal flow depth computation in rectangular-shaped open channels. The convergence of the series solution is shown, and a simple combined model solution is proposed for the truncation of the series with a high degree of accuracy for practical use. Application examples are also given for the sake of illustration.

PROBLEM DEFINITION

For a rectangular channel of a width b conveying a free surface flow under a bottom slope S_0 and a normal depth y_n (Fig.1), the discharge Q is given by the condition of equality between the gravity and friction forces. This condition is expressed by Manning's formula as follows (Chow, 1959):

$$Q = \frac{1}{n} A R^{2/3} \sqrt{S_0} \tag{1}$$

where n is Manning's roughness coefficient, A and $R = A/P$ are the cross-sectional area and hydraulic radius, respectively, and P denotes the wetted perimeter under the normal flow regime. From geometrical considerations, A and R for a rectangular-shaped channel are given by:

$$A = b y_n \tag{2}$$

$$R = \frac{by_n}{b + 2y_n} \tag{3}$$

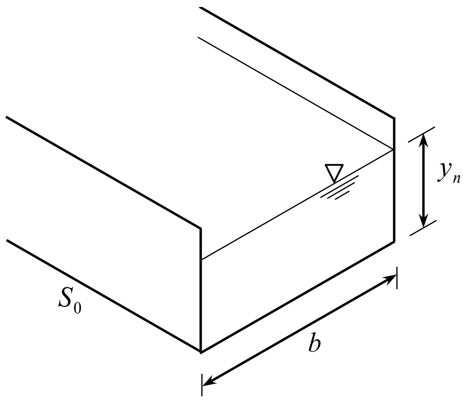

Figure 1: Definition sketch

Inserting Eqs. (2) and (3) into Eq. (1) gives

$$Q = \frac{\sqrt{S_0}}{n} \frac{(by_n)^{5/3}}{(b + 2y_n)^{2/3}} \tag{4}$$

Introducing the dimensionless parameters η and β such that:

$$\eta = \frac{y_n}{b} \quad \text{and} \quad \beta = \frac{nQ}{b^{8/3}\sqrt{S_0}} \tag{5}$$

Eq. (4) is rewritten under a compact form as follows:

$$\beta = \frac{\eta^{5/3}}{(1 + 2\eta)^{2/3}} \tag{6}$$

Eq. (6), allowing the determination of the relative normal depth η, is implicit, and its solution requires an iterative process of trial and error. For an analytical treatment using the proposed δ-perturbation technique, one can rearrange Eq. (6) in the form $x = f(x)$ as:

$$\eta = \beta^{3/5}(1 + 2\eta)^{2/5} \tag{7}$$

Under the present form (Eq.7), most numerical algorithms and analytical solutions attempt to tackle the problem of normal depth computation in a rectangular-shaped channel.

DELTA-PERTURBATION SOLUTION

To solve Eq. (7), which is obviously of an implicit type, we make use of the δ-expansion method. The principle consists, in simple words, of expanding in powers of a nonlinearity present in the governing equation and determining terms of the series expansion by recurrence. This procedure was first introduced by Bender et al. (1989) for nonlinear differential equations. To derive an analytical solution to the problem, let us introduce a small parameter δ in the exponent of the nonlinear term in Eq. (7):

$$\eta - \beta^{3/5}(1+2\eta)^{\delta} = 0 \tag{8}$$

The analytical solution is then sought as an expansion for $\eta(\delta)$ in terms of a perturbation series of δ as follows:

$$\eta(\delta) = \sum_{n=0}^{\infty} a_n \delta^n = a_0 + a_1\delta + a_2\delta^2 + a_3\delta^3 + \cdots \tag{9}$$

On the other hand, the second term of Eq. (8) can be expressed as

$$(1+2\eta)^{\delta} = \exp[\delta \ln(1+2\eta)] \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) and equating the coefficients of equal powers of the perturbation parameter δ, the expressions of the a_n terms are easily determined, and the first three are:

$$a_0 = \beta^{3/5} \tag{11}$$

$$a_1 = \beta^{3/5} \ln\left(1+2\beta^{3/5}\right) \tag{12}$$

$$a_2 = \frac{\beta^{3/5} \ln\left(1+2\beta^{3/5}\right)\left[2\beta^{3/5} \ln\left(1+2\beta^{3/5}\right) + 4\beta^{3/5} + \ln\left(1+2\beta^{3/5}\right)\right]}{4\beta^{3/5}+2} \tag{13}$$

In deriving the expressions of the a_n terms, the following Maclaurin series expansions were used:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{14}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad |x| < 1 \tag{15}$$

Reporting expressions of a_n into Eq. (9), up to a suitable series order, and setting up the perturbation parameter $\delta = 2/5$, the explicit solution of Eq. (7) is then obtained. By introducing, for example, the first two terms only, the explicit solution (Eq. 9) reads:

$$\eta = \beta^{3/5} + \frac{2}{5}\beta^{3/5}\ln\left(1+2\beta^{3/5}\right) + \frac{2}{25}a_2 + \cdots \quad (16)$$

Once the dimensionless flow depth η is computed to a suitable order, the normal depth y_n is then deduced from Eq. (5).

A comparison of the influence of the number of terms considered in the series expansion (Eq. 9) up to $n=6$ as a function of the dimensionless discharge β with the exact numerical solution of Eq. (7) is reported in Fig. (2). One can see from the figure that the flow carrying curve $\beta = f(\eta)$ converges regularly to the actual solution. For a wide channel, as $\eta \to 0$ (or $\beta \to 0$) the solution is given by only the first term of Eq. (16). This corresponds to the 0^{th} order solution, where the friction forces are dominated by the bottom resistance and the sidewall influence can be neglected.

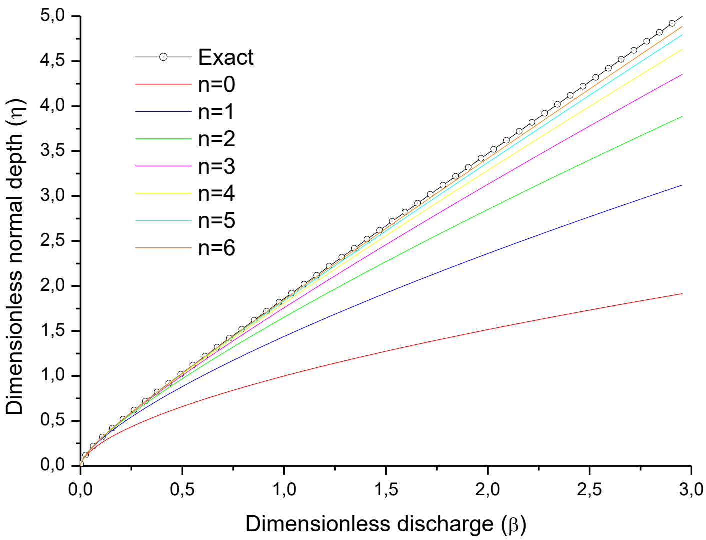

Figure 2: Influence of the number of terms in the series expansion on the convergence of the δ - perturbation solution

In addition to the convergence illustrated graphically in Fig. (2), it can be shown that the solution series in Eq. (16) converges monotonically to the exact solution. Because of the quick vanishing of the δ^n factors more than the increase in the a_n terms, eventual divergence of the series is prevented, and the solution is thus bounded. A more rigorous proof can easily be set up by using the ratio convergence test (Kreyszig, 1979) and considering the parameter $\rho = \lim_{n\to\infty}\left|a_{n+1}\delta^{n+1}/a_n\delta^n\right|$. For $\delta = 2/5$, it can be verified that $\rho < 1$, which confirms the convergence of the series in Eq. (16) independently of β.

At this stage, it is interesting to note that when expanding Eq. (16) in the Maclaurin series, i.e., the logarithm function following Eq. (15), as a function of β and rearranging, one obtains:

$$\eta = \beta^{3/5} + \frac{4}{5}\beta^{6/5} + \frac{4}{5^2}\beta^{9/5} - \frac{16}{5^3}\beta^{12/5} + \frac{896}{5^6}\beta^{18/5} + \cdots \tag{17}$$

This expression (Eq. 17) is simply the solution of Eq. (7) given by Lagrange's inversion theorem (Swamee and Rathie, 2004 and Ferro and Sciacca, 2016). Formally, one can show that the terms of the series in Lagrange's expansion can be generated for Eq. (17) as follows (Achour and Bedjaoui, 2006):

$$\eta = \sum_{n=1}^{\infty} \frac{2^{n-1}\beta^{3n/5}\Gamma(2n/5+1)}{\Gamma(n+1)\Gamma(-3n/5+2)} \tag{18}$$

where $\Gamma(x)$ denotes the Gamma function (Zwillinger, 2018). It follows that Lagrange's inversion expansion can be considered a special case of the δ-perturbation series when the latter is expanded in the Maclaurin series.

It is worth emphasizing that while the radius of convergence of the δ-expansion series in Eq. (16) is infinite, the convergence of Lagrange's series in Eq. (18) is ensured only when $\beta < 0.967$ which corresponds to $\eta < 1.807$ (Ferro and Sciacca, 2017). The restricted radius of convergence of Eq. (18) compared to Eq. (16) can easily be understood when examining the condition on the argument in Eq. (15). It follows that the presence of the logarithm function in Eq. (16) prevents the divergence of the series expansion and ensures a monotone convergence.

To quantitatively show the deviation between the actual and δ-perturbation solutions, Fig. (3) illustrates the relative error in η compared to the exact numerical solution of Eq. (7). In the wide practical range of $\eta \in [0,5]$ and for different numbers of terms included in the expansion series, it can be seen that the maximum deviation diminishes as the order of the series increases. For an economically designed rectangular channel, it is well known that η should be equal to $1/2$ or roughly near this value. For this particular value of η, corresponding to $\beta \approx 0.2$, the maximum deviation for the third-order solution (δ^3) is 0.73 % and drops to 0.028 % for a sixth-order approximation (δ^6). From Fig. (3), it follows that the accuracy of the perturbation solution increases monotonically with the number of terms included in the perturbation series.

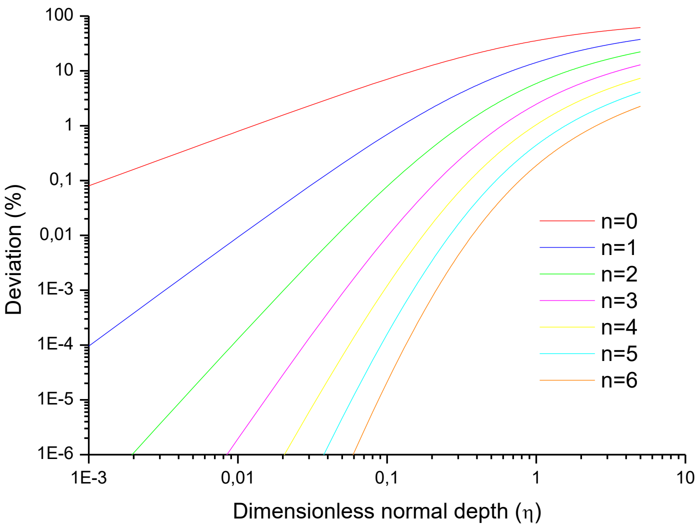

Figure 3: Deviation of the δ-perturbation solution as a function of the number of terms in the series expansion

From a practical standpoint, it is a tedious task to compute terms higher than the second order in Eq. (9) to achieve a desirable accuracy, as might be expected given the increasing size of terms in Eqs. (11) to (13). To render the approach more practical and straightforward, a combined model is constructed from the first two linearized terms of the perturbation series in Eq. (16) and Hoerl's model approximation as follows:

$$\eta = \underbrace{\beta^{3/5} + \frac{4}{5}\beta^{6/5}}_{\text{Terms of the series}} + \underbrace{a\beta^m c^\beta}_{\text{Hoerl's model term}} \tag{19}$$

in which a, m, and c are unknown constants of Hoerl's model. The basic idea of the present combined solution is then to convert the infinite series expansion of higher-order terms into a closed term formed by a mixed power-exponential function approximation. The determination of Hoerl's model constants is carried out by minimizing a residual function $\Re$ defined as:

$$\Re(a,m,c) = a\beta^m c^\beta - \sum_{n=0}^{\infty} a_n \left(\frac{2}{5}\right)^n - \left(\beta^{3/5} + \frac{4}{5}\beta^{6/5}\right) \tag{20}$$

By setting the stationarity condition on the absolute values of the $\Re$ function with respect to the model parameters, one writes:

$$\frac{\partial}{\partial a}\int_0^{\beta_f}|\Re|d\beta = \frac{\partial}{\partial m}\int_0^{\beta_f}|\Re|d\beta = \frac{\partial}{\partial c}\int_0^{\beta_f}|\Re|d\beta = 0 \quad (21)$$

where β_f denotes the upper limit of the dimensionless discharge. The conditions in Eq. (21) translate a weighted residuals concept where a Heaviside's unit step acts as a weighting function. Within the practical range $0 \le \beta \le \beta_f = 2.956$, corresponding to $0 \le \eta \le 5$, the computation of the constants a, m and c satisfying Eq. (21) has led to the following explicit analytical expression for η:

$$\eta = \beta^{3/5} + \frac{4}{5}\beta^{6/5} + \frac{12}{5^3}\beta^{1.641}\left(\frac{171}{271}\right)^{\beta} \quad (22)$$

The deviation of the combined solution model (Eq. 22) from the actual solution of Eq. (7) is reported in Fig. (4). As shown in the figure, the use of Eq. (22) leads to a maximum relative error of only 0.029 % compared to the actual solution of Eq. (7).

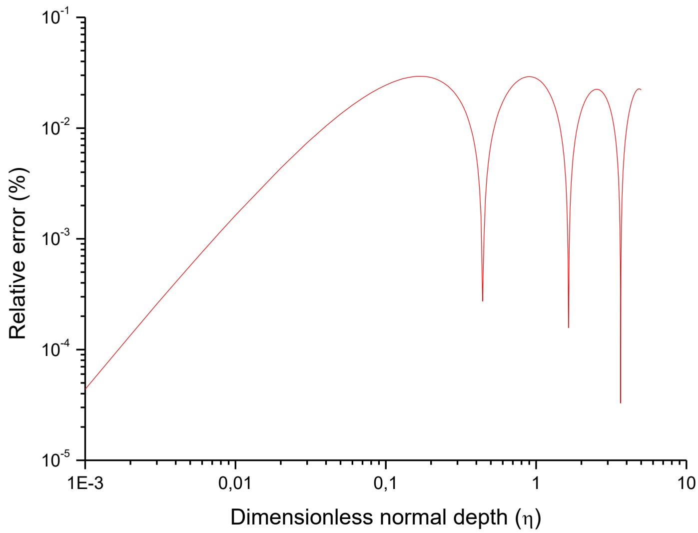

Figure 4: Relative error of the combined model (perturbation-Hoerl) solution (Eq. 22) vs. the nondimensional normal depth

PRACTICAL APPLICATIONS

To illustrate the computation of the dimensionless normal depth using the δ-perturbation expansion (Eq. 9) and the combined model solution (Eq. 22), examples of channels are treated in Table 1, where the geometric and hydraulic properties of the examined channels are summarized. Details of the computed values of η for the different channels compared to the actual solution (by trial and error procedure) are also reported in Table 1.

The results show the accuracy of the perturbation solution to the sixth-order and that of the combined model where the maximum deviation does not exceed 0.1 % and 0.029 %, respectively. The solution issued from the combined model shows a clear advantage compared to the sixth-order series expansion. In the practice of hydraulic engineering, this degree of accuracy is very high.

Table 1: Channel parameters and details of the computation of the dimensionless normal flow depth.

Channel	*1*	*2*	*3*
b (m)	3	3.6	2
n ($m^{-1/3}s$)	0.015	0.025	0.013
S_0 (-)	0.005	0.00025	0.001
Q (m^3/s)	12.0	4.25	6.20
β *[Eq. (5)]*	0.135976594	0.220742318	0.401410184
η *[Eq. (7)]*	0.378417155	0.54182851	0.863938813
$\eta^{(6ord)}$ *[Eq. (9)]*	0.378377018	0.541633781	0.86279646
Error (%)	1.06×10^{-2}	3.59×10^{-2}	1.32×10^{-1}
η *[Eq. (22)]*	0.378449675	0.541764722	0.863688149
Error (%)	8.59×10^{-3}	1.18×10^{-2}	2.90×10^{-2}

It is interesting to keep in mind that even an accuracy of ±0.5 % is well within the degree of physical significance that can be attached to the assessment of the value of Manning's coefficient *n*. The difficulties involved in specifying exact values for *Q* and S_0 are also worth mentioning.

CONCLUSION

A new exact analytical solution for the direct explicit computation of the normal flow depth in rectangular-shaped channels is presented in this paper. Based on the δ-perturbation method, an exact solution was obtained in the form of a series expansion, where the first three terms of the series were given for the sake of demonstration. It was then possible to obtain a general analytical solution, from which Lagrange's inversion solution is deduced as a special case. The accuracy obtained for the sixth-order expansion is of a high level, and the series order can be chosen depending on the required accuracy degree.

The proposed solution (Eq. 9) generates a highly accurate prediction in the practical range of the dimensionless parameter $\eta \in [0,5]$. An arbitrary level of accuracy can be achieved by including more terms in the convergent series expansion. To avoid the need for higher-order terms in the series expansion for practical purposes, Hoerl's model was introduced as a corrective term for the truncation error in the δ-perturbation expansion series. The resulting combined model solution generates a maximum deviation of only 0.029 %, which forms an excellent accuracy in the wide range of $\eta \in [0,5]$. It is interesting to note that, due to its versatility and convergence peculiarities, the present direct δ-perturbation approach, or the combined model, could easily be applied to other cross-section profiles.

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