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NUMERICAL SIMULATION OF TWO-DIMENSIONAL SUPERCRITICAL FLOWS USING A PREDICTOR-CORRECTOR FINITE DIFFERENCES SCHEME

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ABSTRACT

The phenomenon treated in the present research work relates to unsteady two-dimensional free-surface flows in a torrential regime. These flows are governed by Saint-Venant equations which are obtained from the depth integration of the three-dimensional Navier-Stokes equations, taking into account some assumptions in order to simplify the mathematical model governing the type of flow studied. The determined equations cannot be solved theoretically by known algebraic methods. Several numerical methods have been developed to date to solve partial differential equations. In the present study, we opt for the finite difference method. The discretization of the governing equations is done using a Gabutti Predictor-Corrector scheme. A numerical model is developed in order to determine the flow network in a symmetrical transition zone of a free surface channel. It is recalled that this type of structure is widely used in hydraulics, mainly in spillways chutes. These transitions can be either expansions or contractions. In this case, we propose to analyse the flow of water through a hydraulic structure composed of a gradual expansion. The knowledge of the wave system is very important in order to properly dimension this kind of hydraulic structures. The aim is to calculate the water line along the median axis and the sidewall, then move on to a parametric study in which the influence of varying the Manning's roughness number and the upstream width of the channel expansion on the shape of the water line will be analysed.

Key words: 2D unsteady flow, Torrential regime, Saint-Venant, Shock wave, finite differences

INTRODUCTION

A very large number of problems in mathematical physics can be modelled by partial differential equations. The computer tool is not innocent of the interest in modelling. Indeed, almost all phenomena in physics are modelled by evolutionary equations and most often non-linear ones. A manual solution is simply unthinkable. The search for the exact solution is almost unthinkable and only the use of so-called numerical methods remains, of approximations, to hope to understand, in a more or less distant way, what is happening in the system that we wish to describe.

The complex phenomenon treated in the present research work relates to unsteady two-dimensional free-surface flows in a torrential regime. These flows are governed by a non-linear partial differential equations system of hyperbolic type given by Saint-Venant. This system was obtained from the depth integration of the three-dimensional Navier-Stokes equations taking into account some assumptions in order to simplify the mathematical model governing the type of flow studied. Despite these simplifications, the determined equations cannot be solved theoretically by known algebraic methods (Bhallamdi and Chaudhry, 1992; Fennema and Chaudhry, 1990; Ikni et al., 2021; Berreksi et al., 2022).

Several numerical methods have been developed to date to solve partial differential equations including the non-linear hyperbolic equations. The choice of a numerical solution method is rather delicate. In the present study, we opt for the finite differences method. The discretization of the governing equations is done using a Predictor-Corrector type scheme (Gabutti, 1983; Fennema and Chahdhry, 1986; Berreksi et al., 2008; Berreksi et al., 2016).

A numerical model is developed in order to determine the shape of the water line in a symmetrical transition zone of a free surface channel. It is recalled that this type of structure is widely used in hydraulics, mainly in spillways chutes. These transitions can be either expansions and/or contractions. It is also pointed out that these structures are often the site of rather dangerous transverse waves. The knowledge of the wave system is very important in order to properly dimension this kind of hydraulic structures and minimize the occurrence of the so-called shock waves (Rouse et al., 1951; Hager, 1992; Benmebarek and Berreksi, 2018).

In the present study, we propose to analyse the flow of water through a hydraulic structure composed by a symmetrical gradual expansion. The objective is first to determine the shape of the water line and then to study the variation effect of the roughness number and the upstream width on the profile of the free surface in the hydraulic structure studied. The explicit Gabutti finite differences scheme is used to solve the 2D Saint-Venant equations governing the unsteady free surface flow.

GOVERNING EQUATIONS

The unsteady flows in torrential regime are often governed by the Saint-Venant system. The latter is generalised in order to take into account the two-dimensional character of the studied phenomenon. The establishment of the equations of motion is always based on some simplifying assumptions, in particular: (i) hydrostatic distribution of pressures, (ii) uniform distribution of velocities on a vertical and (iii) low slope of the channel bottom. In terms of primitive flow variables *h*, *u* and *v*, the governing equations may be written in Cartesian coordinates (Rahman and Chaudhry, 1997; Benmebarek and Berreksi, 2018; Ikni et al., 2021; Berreksi et al., 2022) as: $V_t' = (h, u, v)^t$, where:

$$V_t' + P_x + R_y + T = 0 \tag{1}$$

in which:

$$P_x = \begin{bmatrix} uh \\ \frac{u^2}{2} + gh \\ uv \end{bmatrix}; \qquad R_y = \begin{bmatrix} vh \\ uv \\ \frac{v^2}{2} + gh \end{bmatrix}; \qquad T = \begin{bmatrix} 0 \\ -g(S_{0x} - S_{fx}) \\ -g(S_{oy} - S_{fy}) \end{bmatrix} \tag{2}$$

where *h* is the flow depth; *u* the depth averaged flow velocity in the *x* direction; *v* the depth averaged flow velocity in the *y* direction; *t* the time; *g* the acceleration due to gravity; S_{ox} and S_{oy} the channel bottom slopes in the *x* and *y* directions respectively; S_{fx} and S_{fy} the friction slopes in the *x* and *y* directions respectively.

The bottom slopes are calculated as follow (Berreksi et al., 2008; Berreksi et al., 2016):

$$S_{ox} = \sin \alpha_x \tag{3}$$

$$S_{oy} = \sin \alpha_y \tag{4}$$

where α_x and α_y are the incline angle of the channel bottom in the *x* and *y* directions respectively.

The friction slopes are calculated as follow:

$$S_{fx} = \frac{n^2 u \sqrt{u^2 + v^2}}{h} \left(\frac{b+h}{bh}\right)^{1/3} \tag{5}$$

$$S_{fy} = \frac{n^2 v \sqrt{u^2 + v^2}}{h} \left(\frac{b+h}{bh}\right)^{1/3} \tag{6}$$

where *n* is the Manning roughness coefficient and *b* the channel width.

WATER FLOW IN A CANAL EXPANSION

A dam has two important components: the spillway and the energy dissipator. The spillway controls overflow, particularly during floods, and conveys the water downstream of the dam. The energy dissipator receives the spilled flow at the base of the dam and returns it safely to the river or lake. The connection between the spillway weir and the dissipater is ensured by means of appropriate devices. It should be noted (Boes and Hager, 2003; Wu, 2008) that due to the large differences in level between the spillway weir and the river bed, the kinetic energy at the base of the dam can be enormous and the velocities, as a result, will be quite considerable.

In practice, channel expansions are frequently found in torrential flows where the flow exits at high velocity from a closed pipe, a bottom gate, a spillway or a steep weir (Rouse et al., 1951). Figure 1 illustrates a gradual channel expansion (Hager, 1992).

In a transition with diverging walls, the flow velocity increases, the water depth decreases, the shock angle βi increases and the transverse waves diverge accordingly. Therefore, such expansions do not generate abrupt variations in water height.

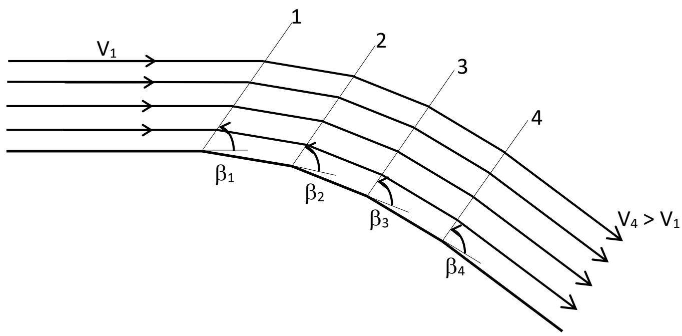

Figure 1: Transverse waves due to a concave wall

NUMERICAL RESOLUTION OF THE GOVERNING EQUATIONS

The unsteady free surface flows are governed by a system of nonlinear hyperbolic partial differential equations. Such equations can be solved theoretically only in particular cases not easily found in reality. Thus, several problems in hydraulics require for lack of an analytical solution, a numerical solution of the partial differential equations. One of the classical methods to approach this solution is the use of the finite differences method.

In the current case, the Gabutti explicit finite differences scheme which is a second order accurate in space and time is chosen (Gabutti, 1983; Fennema and Chaudhry, 1986; Fennema and Chaudhry, 1990). In this scheme, a two-part Predictor step and a single Corrector step are used with the spatial derivatives. The solution at the unknown time level is calculated by using the obtained results in the Predictor's and the Corrector steps.

This scheme uses the equations of motion written essentially in non-conservative form. The non-conservative form of the system of equations given by equation (1) is:

$$V'_t + G'^+ V'^+_x + G'^- V'^-_x + H'^+ V'^+_y + H'^- V'^-_y + T = 0 \tag{7}$$

In which:

$$P_x = G'V'_x \ (P_y = G'V'_y) \ ; \quad R_y = H'V'_y$$

G' and H' are Jacobian matrices. These two matrices are also decomposed into two matrices; the first contains positive values, while the second contains negative values.

Applying the Gabutti scheme formulation to the equations of motion of the studied phenomenon (Eq. 7) gives:

Prediction step (Part I)

$$\begin{aligned} \tilde{V}'_{i,j} = V'^k_{i,j} &- \tau_x \left[G'^{+k} \nabla_x V'^k_{i,j} + G'^{-k} \Delta_x V'^k_{i,j}\right] \\ &- \tau_y \left[H'^{+k} \nabla_y V'^k_{i,j} + H'^{-k} \Delta_y V'^k_{i,j}\right] - \Delta t\, T^k_{i,j} \end{aligned} \tag{8}$$

Prediction step (Part II)

$$\begin{aligned} \overline{V}'_{i,j} = V'^k_{i,j} &- \tau_x \left[G'^{+k} \left(1 + \nabla_x\right) \nabla_x V'^k_{i,j} + G'^{-k} \left(1 - \Delta_x\right) \Delta_x V'^k_{i,j}\right] \\ &- \tau_y \left[H'^{+k} \left(1 + \nabla_y\right) \nabla_y V'^k_{i,j} + H'^{-k} \left(1 - \Delta_y\right) \Delta_y V'^k_{i,j}\right] - \Delta t\, T^k_{i,j} \end{aligned} \tag{9}$$

Correction step

$$\begin{aligned} \hat{V}'_{i,j} = \tilde{V}'_{i,j} &- \tau_x \left[\tilde{G}'^+ \nabla_x \tilde{V}'_{i,j} + \tilde{G}'^- \Delta_x \tilde{V}'_{i,j}\right] \\ &- \tau_y \left[\tilde{H}'^+ \nabla_y \tilde{V}'_{i,j} + \tilde{H}'^- \Delta_y \tilde{V}'_{i,j}\right] - \Delta t\, \tilde{T}_{i,j} \end{aligned} \tag{10}$$

Where $\nabla_{x,y}$ and $\Delta_{x,y}$ denote respectively a backward finite difference and a forward finite difference along x and y directions.

The new value of V' at time step (k+1) is calculated by:

$$V'^{k+1}_{i,j} = \frac{1}{2}\left(V'^k_{i,j} + \overline{V}'_{i,j} + \hat{V}'_{i,j} - \tilde{V}'_{i,j}\right) \tag{11}$$

STABILITY CONDITION

Finite difference techniques are commonly employed in solving numerically partial differential equations. The obtained solutions from explicit finite difference schemes are conditionally stable where the stability condition is given by the Courant-Friedrichs-

Lewy (CFL) restriction. For two dimensional flows, this condition is given by the following equation (Bhallamudi and Chaudhry, 1992; Berreksi et al., 2008):

$$C_n = \frac{\left(V + \sqrt{g h}\right)\Delta t}{b(x)\,\Delta x\,\Delta y}\sqrt{\Delta x^2 + \left[b(x)\,\Delta y\right]^2} \qquad (12)$$

where, V is the resultant velocity at the grid point; C_n is the Courant number; Δx and Δy are the distance increment in x axis and y axis respectively; Δt is the time interval. For the Gabutti schemes, the value of Courant number must be inferior or equal to two ($C_n \leq 2$).

STUDY CASE

Symmetrical expansion

We propose to study the case of a symmetrical gradual expansion crossed by a high velocity flow. The main dimensions of the said structure are (Rouse et al., 1951; Bhallamudi and Chaudhry, 1992): a length of 0.549 m; a "depth-width" ratio of 0.25; an upstream Froude number equal to 2; the channel is horizontal; a Manning's roughness coefficient of 0.012; an upstream water height of 0.0305 m; a longitudinal upstream velocity of 1.094 m/s. The objective is to determine the water line profiles along the median axis and the side wall of the enlarged structure. The mesh considered is Δx=0.0483m and Δy = 0.0476 m. The CFL number is equal to 0.8.

Figures 2 and 3 show the obtained profiles compared to experimental measurements existing in the specialised literature in the present field (Bhallamudi and Chaudhry, 1992).

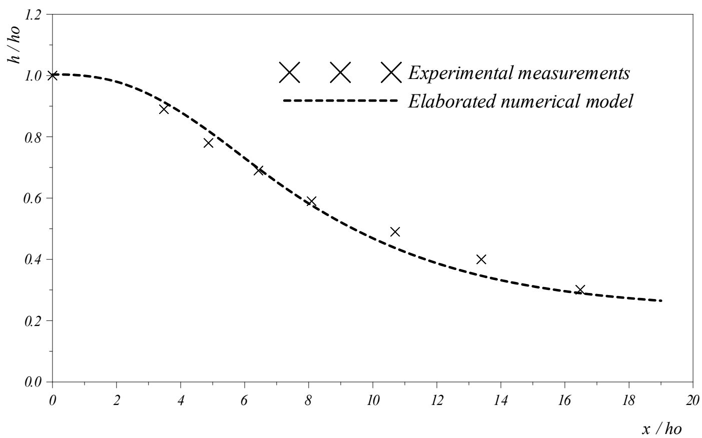

Figure 2: Water line profile along the median axis (channel expansion)

From the Figure 2, it is thus found that the water line decreases in a progressive manner with the length of the transition. There is no sign of a transverse wave system appearing across the hydraulic structure studied. The obtained results are in very good agreement with the experimental measurements made by other researchers.

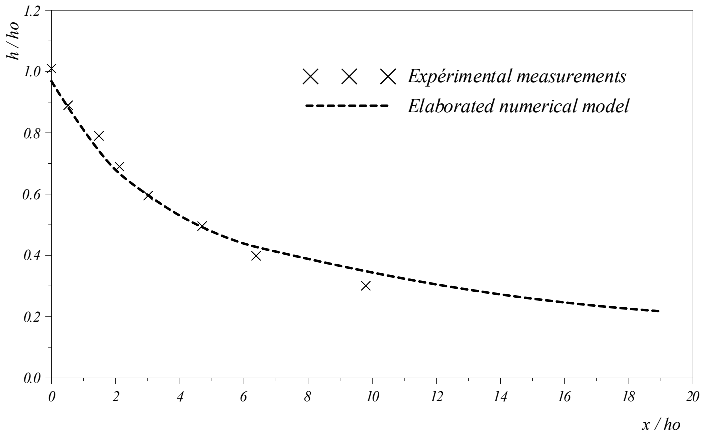

Figure 3: Water line profile along the side wall (channel expansion)

From the Figure 3, it is noticed that the water line along the side wall decreases with distance in a more pronounced way compared to that of the median axis. The flow is also without any disturbance or undulation at the surface. A good agreement is observed here between the numerical results compared to the experimental ones of other researchers.

It can be said that the obtained results in this part are very satisfactory and presents a very appreciable approach. The numerical model developed is very successful in simulating high velocity flows through a gradual symmetrical expansion.

Effect of varying Manning's number

The consequence of the variation of the Manning's roughness coefficient on the water line shape is discussed. The obtained results with Manning's number values of 0, 0.014, 0.016 and 0.018, compared with those calculated with the reference value of 0.012, are shown in Figures 4 and 5.

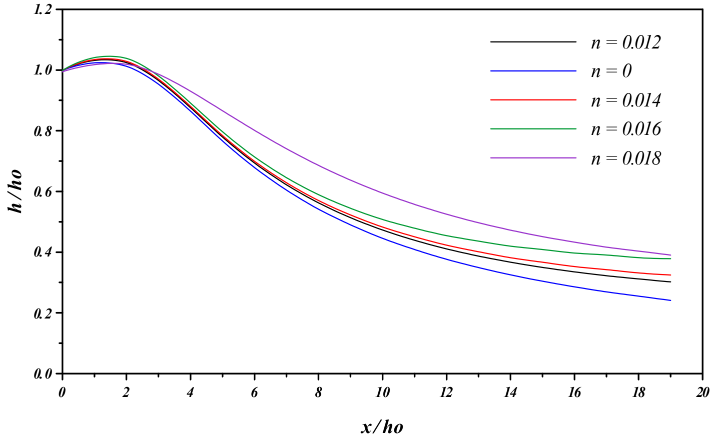

Figure 4: Variation of the Manning's number in a gradual (Median axis)

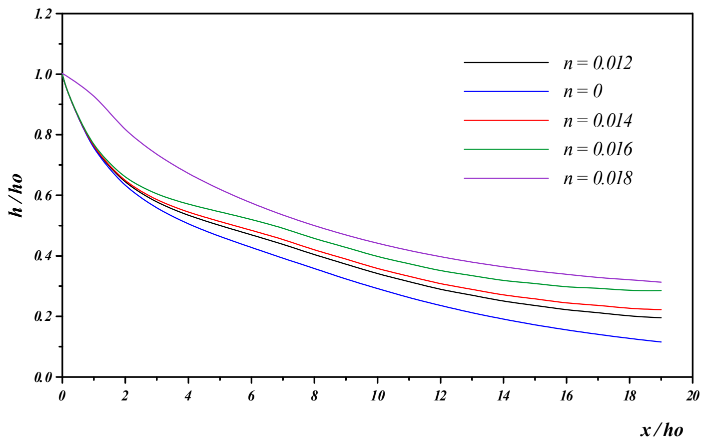

Figure 5: Variation of the Manning's number in a gradual (Sidewall)

It can be seen that the water line at the axis and along the wall has the same overall shape. As the value of this coefficient increases, the water line will be slightly higher from the middle of the hydraulic transition. If the value is lower than the reference one, the water line will be lowered. The rise in the water profile becomes significant at a value of 0.018.

Effect of varying upstream width of the transition

The water line profiles along the axis of symmetry and along the wall, shown in figures (6) and (7) below, are calculated for upstream widths of 0.110 m, 0.115 m, 0.130 m and 0.150 m, in addition to the reference value of 0.122 m.

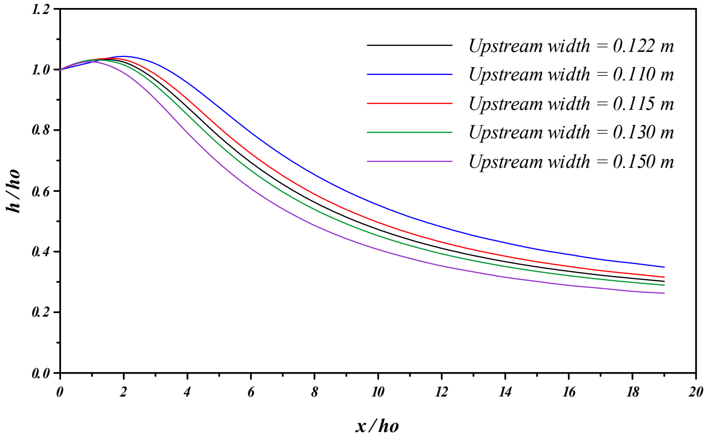

Figure 6: Influence of varying upstream width of the channel expansion on the water surface (Median axis)

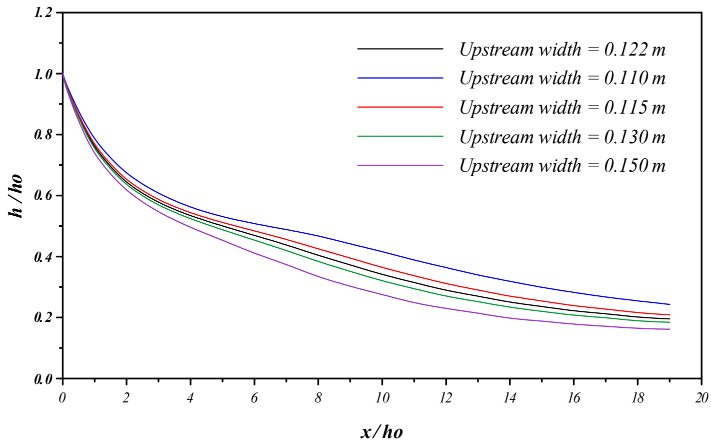

Figure 7: Influence of varying upstream width of the channel expansion on the water surface (Sidewall)

From these figures, we can see that all the curves follow the same shape. The water lines decrease from upstream to downstream, but they start at the same point $h=h_0$ (same condition for the upstream height). The variation in water height along the expansion is inversely proportional to the width b. The water line corresponding to a given width b will be below a water line corresponding to a smaller width b, and vice versa.

By increasing the upstream width b of the transition, the whole dimension of the progressive expansion changes; and with the same upstream boundary conditions, the increase in width b leads to a spreading of the water depth, which will create a decrease in the water height h.

CONCLUSION

The objective of the present study is the numerical analysis of high velocity free surface flows. A numerical model is developed and tested for the case of a gradual channel expansion in supercritical flow. The water line profiles at the centreline and the sidewall were calculated and compared with existing experimental results in the literature related to the present research area. A very good agreement is observed both at the symmetrical axis and the sidewall. Therefore, it can be clearly concluded that the numerical model carried out is suitable to simulate the type of phenomenon treated. It should be noted that in such a hydraulic structure no disturbance or agitation in the free surface is observed, and consequently the amplitude of the transverse waves is very low if not non-existent. Indeed, the increase in the angle of deflection of the wall expansion is sufficiently gradual and therefore does not cause any abrupt change in the flow depth at any section of the channel. On the other hand, the effect of varying the Manning's roughness number on the flow is studied. Thus, it is found that the change of this number, which is directly related to the roughness, has an influence on the variation of the water level in the gradual expansion studied. Each time this number is increased, an increase in water level is observed. This is due to the slowing down of the flow caused by the increased friction. Furthermore, varying the upstream width of the transition also influences the shape of the water line. Furthermore, varying the upstream width of the transition also influences the shape of the water line. In fact, by increasing the upstream width of the channel expansion, the whole dimension of the gradual expansion changes. The variation in water height along the expansion is inversely proportional to the upstream channel width.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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