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MODIFIED MONTANA FLUME (MMF)

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ABSTRACT

Flow measurement has become essential in the field of water engineering. Accuracy is required in various fields, such as industrial, municipal, and agricultural effluents. It allows better knowing and sizing the equipment of water supply, water collection or treatment works, or even to better know the quality of water bodies and to quantify the availability of water resources. In open channels, the measurement of flow rates would enable knowing with great precision the evacuation capacity of the structure. For irrigation needs, part of this water is diverted, and the plot to be irrigated requires a quantity of water that must be assessed with the greatest exactness.

For open channel flow measurement, flumes are the most commonly used structures, the best known of which are the Parshall and the Venturi. The principle of these devices is based on a lateral contraction of the walls, sometimes on a localized elevation of the bottom, or else on both. As a general rule, they are formed of three static parts, namely, a converging part as the first part, followed by a canal of a constant section called the neck or throat as the second part, and finally, a terminal divergent part at the outlet of the device called the discharge section as the third part. All cross-section shapes of the device are mostly rectangular. The throat acts as a control section, where flow is critical, allowing the device to produce a relationship between the upstream water level and the flow rate, also called the stage-discharge relationship. Thus, the flow rate sought is deduced as soon as the depth of the upstream flow is measured.

Montana Flume, which is the subject of investigations during this study, is less well known than the aforementioned two devices, although it has certain advantages. It is a truncated version of the Parshall since it is only formed by a flat-floored converging part;

it has no throat or discharge section. As a result, it takes up less space and is less expensive. As with most flumes, the Montana flume is an empirical device on which tests were performed to derive the empirical stage-discharge relationship. Under downstream free-flow conditions, the stage-discharge relationship is expressed as $Q = K\,h^n$, where Q is the discharge, h is the upstream stage, K is the flume discharge constant depending on the flume size, and n is the discharge exponent depending also on the flume size, as reported in the specialized literature.

It is not the form of the previous stage-discharge relationship that is disputed by the authors of the present study, but it is the fact that the exponent n varies from one device to another according to their size. The variation in the exponent n inevitably leads to a change in the dimensions of the constant K, which does not conform to the proven principles of flow measurement. Additionally, the exponent n takes values varying between 1.522 and 1.566 for devices from 1-inch to 36-inch in size, while it should be equal to 1.5 given the involved rectangular cross-section shape.

The main objective of the present study is to give more rationality to the stage-discharge relationship of Montana flumes, derived from a convincing theoretical development based on simplifying hypotheses with reduced effect. The resulting stage-discharge relationship takes the accepted form of weirs, where the exponent n is equal to 1.5 and does not change with device size. However, the rationality thus expected results in modifying one of the linear dimensions of the original Montana flume, in particular the width of the outlet section; this leads to suggest modified Montana flumes that are more efficient, requiring less space. Based on the very wide range of experimental discharges and measured upstream depths provided by the literature, the modified Montana flume is characterized by an optimal contraction rate deduced from the optimization of the theoretical stage-discharge relationship. Moreover, the theoretical stage-discharge relationship, corrected for the effects of a given constant and the relative upstream depth related to the channel approach width, causes deviations in flows rates computation often lower than those inferred by using the original Montana flumes. In addition, the optimization carried out on the device induces smaller linear dimensions than those of the original Montana flume, thus requiring less material for its design. Finally, the authors recommend a relevant approach for appropriate sizing of the advocated flume.

Keywords: Flow measurement, Discharge, Theoretical analysis, Discharge coefficient, Stage-discharge relationship, Montana flume.

INTRODUCTION

The most preferred approaches for measuring discharge in open channels are the so-called "direct discharge methods" (Bos, 1989; Achour et al., 2003). These methods do not require measurement of the flow velocity. They use certain devices for the direct measurement of the flow sought through a relationship, often empirical, called the stage-discharge relationship. By introducing the measured upstream depth into this relationship, the corresponding flow rate is determined. These methods involve different types of devices, such as weirs and flumes that are of interest to the current study.

Among the simplest weirs are thin plates made of notches of various shapes. For the first time, this type of weirs has recently been revisited from a theoretical point of view, with the main objective of deducing the relationship that governs the discharge coefficient. This was made possible with the help of the energy equation transformed into dimensionless terms, assuming a localized control section above the weir crest. The theoretical discharge coefficient relationship thus derived was then corrected to give results consistent with reliable observations available in the literature. The studies concerned weirs of all known notch shapes, such as contracted or suppressed rectangular shape (Achour et Amara, 2021a), parabolic in shape which has been studied both from a theoretical and experimental point of view (Achour and Amara, 2021b), circular in shape (Amara and Achour, 2021), triangular in shape whose the derived theoretical relationship governing the discharge coefficient has been corroborated by an in-depth experimental study including the special case of the 90° V-notch (Achour and Amara, 2021c; Achour and Amara, 2021d). The triangular shape is shown to be the most accurate in flow measurement, for both high and low flow rates. This is due to the perfect geometric similarity that characterizes such a form.

The optimal dimensions of the previously mentioned reduced thickness weirs must meet requirements to ensure their proper functioning (SIA, 1936; Achour et al., 2003). In addition, they require regular cleaning as these weirs are not self-cleaning so they provide a barrier in front of which solid debris can accumulate. Thus, these weirs must be cleaned periodically, contributing to increasing maintenance costs.

Flow rate can also be measured using finite-length weirs which extend in the direction of the flow. These weirs are called broad-crested weirs provided that their length, related to the upstream flow depth, meets well-defined requirements (Achour et al., 2003). These weirs may be devoid of crest height, making them self-cleaning. The best known and best investigated shapes, both from a theoretical and experimental point of view, are the rectangular and triangular shapes. The main objective of the theoretical development was to deduce the discharge coefficient relationship, based on the well-known fundamental principles of hydraulics. The theoretical relationship thus derived has been corroborated by intense reliable observations (Achour and Amara, 2022a; 2022b), allowing the user to calculate the sought flow rate with both insurance and high accuracy.

Recently, other forms of broad-crested weirs have emerged. Indeed, the papers of Kulkarni and Hinge (2021; 2023) report the computational fluid dynamics and experimental fluid dynamics studies conducted for the measurement of release by a composite broad-crested (CBC) weir and by the manufacturing compound weirs, for exact discharge measurement as rightly stated by the authors.

Afterwards, Achour and Amara (2021e) studied the possibility of measuring the discharge in a rectangular open channel using a sharp-edged width constriction. An elaborate theory allowed the authors to derive the relationship governing the discharge coefficient, whose reliability and accuracy have been confirmed and corroborated by intense qualitative observations. Moreover, the authors state that the sharp-edged width constriction is undoubtedly the simplest and most efficient measuring device, with an easy design and implementation that also requires minimal space.

Between the thin-plate weirs and broad-crested weirs categories, mentioned above, there is the weirs category characterized by a triangular longitudinal profile. These are weirs that can also be an effective means for measuring the flow rate sought. The best-known of these devices and the oldest in their category are the Bazin and Crump weirs. However, the Bazin weir did not meet with enthusiasm from users because it was calibrated for a crest height of 50 cm, which is too high to be used in existing hydraulic installations. Since then, no studies have been undertaken to correct this drawback. The major drawback of both devices is that they can only be used in rectangular open channels so they are not of universal range. In addition, their rectangular shape does not ensure satisfactory accuracy for low flow rates.

Designed and tested by the authors, a new type of weir also belonging to the triangular longitudinal profile category was recently suggested. This is the 2A triangular weir (Achour and Amara, 2023a). Unlike the two previously mentioned weirs, the 2A triangular weir is entirely characterized by triangular-shaped cross-sections, which ranks it among the weirs of great accuracy. This has been demonstrated by the rigorous analysis of observations. It is worth noting that the 2A triangular weir has the same upstream and downstream slopes as the Crump weir. These slopes were adopted because they ensure flow without any disturbance and good adherence to the device's walls. The discharge coefficient relationship that governs the weir has been deduced theoretically with the help of the energy equation, corroborated by qualitative laboratory tests. Moreover, this device is of universal range, being able to be used in open channels regardless of the shape.

Extensive investigations have been recently undertaken by the authors concerning the Crump weir to improve the accuracy of the relationship governing the discharge coefficient (Achour and Amara, 2022a). Based on both rigorous theory and the most recent data available in the literature (Zuikov, 2017), the new semi-empirical relationship governing the discharge coefficient is of unrivalled accuracy since it causes a maximum deviation of only 0.864%, thus significantly improved compared to 4.909% as the maximum deviation induced by the relationship proposed in the literature (Zuikov, 2017). Furthermore, the authors showed that the effect of the h/B ratio on the discharge coefficient of the Crump weir is of the average value of 23.5%, where h is the upstream flow depth and B is the width of the rectangular approach channel. It is worth noting that the previous relationships available in the literature have ignored this effect, which causes significant deviations when using them in flow rate and discharge coefficient calculations.

As previously stated, the use of flumes is of the main interest in this study, as a measurement flow rate mean. There is a whole range of flumes, of various sizes and multiple designs. However, the most useful and well-known flumes are hydraulic jump gauges, such as the Parshall flumes (Parshall, 1936) or the Venturi flume both in its original and modified version (Bos, 1989; Hager, 1985). However, their cross-sections are rectangular, causing a low accuracy for shallow depths and low discharges. This is why a new type of hydraulic jump gauge was designed with triangular cross-sections, thus offering better accuracy (Achour, 1989). The appropriate stage-discharge relationship governing the flume has been established, based on both rigorous theory and intense observations.

From a design point of view, flumes are generally designed in three static parts which play an important role. A short flat-floored converging channel constitutes the first part, followed by a constant cross-section channel which is the second part called the throat, and finally a short diverging channel as the third part, often called discharge sections. However, as will be seen later, some flumes are devoid of discharge sections and even of throat.

The operating principle of flumes is based on the gradual lateral contraction of the flow due to the flat-floored converging section where the flow is in a subcritical state. The flow is accelerated before transforming into a supercritical flow downstream of the outlet cross-section through a "critical transition" in a so-called "control section". Into the discharge sections, the flow may be subcritical through a hydraulic jump or by creating artificial submergence. The control section originates in the throat where flow is critical. The presence of a control section is the *sine qua non* condition for the proper functioning of the device as a flow measurement tool (Achour and Amara, 2022b; 2022c).

The change in state of the flow previously described is due to the shape of the flumes, specially designed for this purpose (Bos, 1989). When subcritical flow passes through the flume and changes into critical and supercritical states, it is proven that the upstream flow depth is independent of the downstream flow depth. Any disturbance that may occur downstream does not affect the flow upstream of the device because the supercritical flow plays the role of what could be defined as a "sanitary cordon", in the sense that the supercritical flow slice prevents any downstream disturbance from moving upstream by ascending the liquid stream. In such a flow situation, the discharge is a single-valued function of the upstream depth from a mathematical point of view. This means that each upstream depth belonging to the depth domain maps to a single well-defined discharge of its range. Consequently, the flow rate can be accurately evaluated in the upstream part of the flume by taking a single depth reading at a specific point of measurement. It is this fundamental principle that flumes used in flow measurement and the resulting single-valued functions are well known as the stage-discharge relationships. The protective effect of the supercritical flow slice avoids any submergence, so that the device operates under free-flow conditions, ensuring its semimodular functioning (Achour, 1989). As soon as the protective effect of the supercritical flow slice disappears, downstream disturbances, such as elevation of the water level, reach the upstream flow, and submergence arises. The device is said to be submerged, or the flow is submerged. In this case, the flow rates measured when the device was operating without submergence must be corrected for the effects of the downstream depth. Several studies have been conducted to develop corrections on empirical or theoretical flow rate relationships for submerged devices (Robinson, 1965; Abt et al., 1995; Willeitner et al., 2012; Kumar and Sarangi, 2022).

It was only until the 1970s that other types of flumes emerged. The various existing flumes are well described in the specialized literature (Bos, 1989; Hager, 1986). The most preferred ones that can be mentioned are namely: Replogle-Bos-Clemens (RBC) flumes (Replogle, 1975), which are portable long-throated flumes with a horizontal sill across the width device; cutthroat flumes proposed by Skogerboe et al. (1972), which are devoid of a throat, hence the name "Cutthroat", unlike many other flumes and are simply formed by converging and diverging sections; SM-flume (Samani and Magallanez, 2000) for which studies have focused on the stage-discharge relationship without alluding to the discharge coefficient (Ferro, 2002; Baiamonte and Ferro, 2007; Di Stefano et al., 2008; Vatankhah, 2017; Vatankhah and Mohammadi, 2020). The name SM-flume evolved to become SMBF-flume, where the initials BF are those of the authors Baiamonte and Ferro (2007) who carried out further investigations. Recently a thorough study has focused on the discharge coefficient of the SMBF-flume, denoted C_d (Achour and Amara, 2023b). It has been shown that the discharge coefficient C_d of this device depends on both the device contraction rate, denoted β, and the relative flow depth h/B, where h is the upstream flow depth and B is the rectangular approach channel width. The study mainly revealed the existence of a zone of non-influence of h/B on the discharge coefficient, in which C_d only depends on the contraction rate β; circular flumes (Samani et al., 1991); central baffle flumes, whose theoretical flow rate relationship was derived using dimensional analysis (Ferro, 2016); the calibration of the derived relationship was carried out with the help of the observations of Peruginelli and Bonacci (1997). It is worth noting that many authors, such as Kolavani et al. (2019), Bijankhan and Ferro (2019), and Aniruddha et al. (2020), have experimentally investigated the effect of different geometrical parameters of a central baffle flume; and the curved wall triangular flume (CWTF) developed recently by Achour and De Lapray (2023c), characterized by triangular converging sections and a triangular throat of constant apex angle, offering then the best accuracy. The dimensions of the device were derived from rigorous geometric considerations. In addition, the theoretical discharge coefficient C_d relationship was inferred using two distinct rational methods: one was based on the energy equation transformed into dimensionless terms, and the other exploited the properties of a kinetic factor. Although distinct, both methods yielded the same result. The predicted discharge coefficients were in excellent agreement with the observations since a maximum deviation of only 0.07% was observed, which confirms the expected excellent accuracy of the flume.

A trapezoidal flume is another well-known type of water measurement device, developed in the sixties and widely disseminated throughout the specialized literature (Robinson and Chamberlain, 1960; Robinson, 1966). The primary objective of these authors was to create a device capable of measuring flow rates with great accuracy in a wider range than that of the Parshall flume. In their major study, Ackers and Harrison (1963) gave a full report on the development carried out on trapezoidal flumes at the hydraulics research station of Wallingford. They highlighted foremost that the observation meriting particular attention is that the device calibration curve can be derived, with satisfactory accuracy, from the boundary layer concept involving a drag coefficient. In addition, a varied selection of ten sizes of trapezoidal flumes is available in the literature, which can be used especially where flow rates are variable. However, one of the disadvantages of the suggested trapezoidal flumes is that the free-flow discharge equations, derived from laboratory and field tests, have not been standardized by industry or national standards. In addition, there are as many relationships governing the flow rate as there are devices, meaning that there is no single relationship that governs the flow rate in all existing trapezoidal flumes, depending on the size of the device considered. Furthermore, the form of the stage-discharge relationship does not meet the well-proven flow measurement principles in open channels. Another disadvantage is that the trapezoidal flume is quite bulky and complex to design.

Thus, the previous compelling reasons led the authors Achour et al. (2024) to recently develop a new type of trapezoidal flume capable of accurately measuring the sought flow rate using a single rational stage-discharge relationship derived from rigorous theory, successfully corroborated by relevant observations. The advocated stage-discharge relationship is applicable regardless of the size of the device or the shape of the approach channel, meaning that the device is of universal range. The main reason is that the suggested trapezoidal flume is designed such that there is no transition between the device and the walls of the approach channel. In addition, the authors provided for design device that occupies a minimum of space and whose optimal dimensions have been recommended for suitable functioning.

The formerly developed flumes, such as the previously cited Parshall, Venturi and even Montana, in which our study is interested, are empirical devices for which flow rate experimental investigations were conducted to derive their free-flow stage-discharge relationship. This is given in the following form: $Q = K\,h^n$ where Q is the flow rate, K is the flume discharge constant, h is the measured upstream depth, and n is the discharge exponent. From the theoretical point of view, the constant K is generally presented as a function dependent on the discharge coefficient C_d, corresponding to the ratio of the actual discharge to the ideal discharge, and on a linear dimension L defining the shape of the cross-section of the device as the width B of a rectangular section.

The question of the mathematical form of weirs and channels producing a certain discharge law was treated in the past by Cowgill (1944). It was shown that for a rectangular shape, its rational stage-discharge relationship is given as a function of h^n, where the exponent $n = 3/2$. This result can be obtained with less sophisticated mathematics by considering both the equation of continuity and Torricelli's velocity formula $\sqrt{2gh}$, where g is the acceleration due to gravity. In the same way, one may show that n is equal to 5/2 in the case of a triangular cross-section. Thus, the exponent n depends solely on the shape of the considered section. It is important to specify this fundamental principle because, in many cases, the exponent n in the empirical depth-discharge relationships that govern certain devices takes inconsistent values. This is the case of the Montana flume, for which n varies between 1.522 and 1.607 (Open channel flow, 2024), depending on the size of the device, while the shape of the involved section is rectangular meaning that the exponent n the value should be 1.50. Moreover, the variation in the exponent n systematically involves a change in the dimensions of the constant K, which is contrary to proven physical principles.

In the case of a rectangular section, when the analysis of the experimental results indicates that the exponent n deviates significantly from its legitimate value of 1.5, it is recommended to suspect the influence of the relative upstream depth on the discharge coefficient of the device. This is a basic principle that should be followed to successfully derive the appropriate stage-discharge relationship sought. In the case of the Montana flume, the relative upstream depth would be related to the width of the inlet; this is what the authors plan to consider. However, from an analytical point of view, it is not easy to take into account the possible effects of the relative upstream depth on the discharge coefficient of the device. What should be done, initially, is to analytically establish the relationship that governs the discharge coefficient of the device assuming no influence of the relative upstream depth, then, in a second step, to correct the derived theoretical relationship by the upstream relative depth effects with the help of the significant experimental data available in the literature. This is the right recommendation that the authors will follow.

The authors will put forward a theoretical approach to infer the stage-discharge relationship for the Montana flumes and then will work out that governing the discharge coefficient. The theoretical approach will be based on the judicious manipulation and relevant rearrangement of the energy equation while considering the effects of the approach flow velocity. The theoretical relationship that the authors plan to establish will be characterized by more rationality than that which currently governs each of the Montana flumes available in the literature and whose form has been previously identified. The improved relationship will be put in the form of that usually used for weirs, in which the exponent n is equal to 1.5 regardless of the size of the device.

Nonbinding simplifying hypotheses will be issued as an ease of support for the theoretical development. However, the effects of these assumptions on the final result will be mitigated when correcting the theoretical stage-discharge relationship by the relevant experimental data available in the literature.

By introducing the experimental data into the theoretical stage-discharge relationship and applying an optimization process using inherent mathematical tools, the optimal values of the contraction rate of the device will be deduced. They will be different from those of the original Montana flumes, hence the name "modified Montana flume". This will also feature a different design significantly more economical since it requires fewer materials for its manufacture.

ORIGINAL MONTANA FLUME

Description of the device and the resulting flow characteristics

Originally, Montana flume was composed of a single element represented by a flat-floored converging open tunnel flow of rectangular cross-sections along its entire length, assembled into one piece, as shown in Fig. 1.

Figure 1: Original Montana Flume

Everyone agrees that the design of Montana flume is inspired by the widely used Parshall flume (Parshall, 1936) devoid, however, of both the throat and outlet discharge sections. Montana flume is thus a truncated version of Parshall flume, only constituted by the converging element, whose contraction rate can be defined by the ratio $\beta = b/B$; the linear dimensions B and b are the widths of the inlet and outlet sections of the device, respectively (Fig. 1). In addition, the device is known to be versatile since engineers, having worked in the areas of hydraulic engineering, used it in a diverse number of applications, such as stream gauging and irrigation canals.

The material intended for the construction of the device depends essentially on the nature of the water flowing and of sediments that it contains. For instance, stainless steel is most commonly used when water contains chemicals or abrasive solids.

Given its reduced dimensions compared to other devices of the same utility, such as the Parshall flume, Montana flume is far less expensive. In addition, the planar geometry that characterizes its elements, such as walls and bottom, makes it easy to build.

When constructed in the form shown in Fig. 1, that is, in the form of a one-piece, it can only be used for a given width channel whose width is the same as the inlet section of the device. For another width of the approach channel, a suited flume must be built and sized. For this reason, the device is available in different sizes, more than twenty according to the literature (Open channel flow, 2024). Figs. 2 and 3 show the geometric parameters of the original Montana flume, the plan view of both the channel and the device, and the longitudinal flow profile.

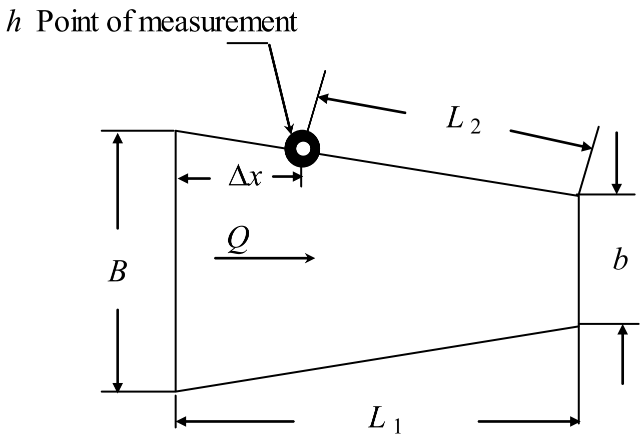


Figure 2: Schematic plan view of an original Montana Flume

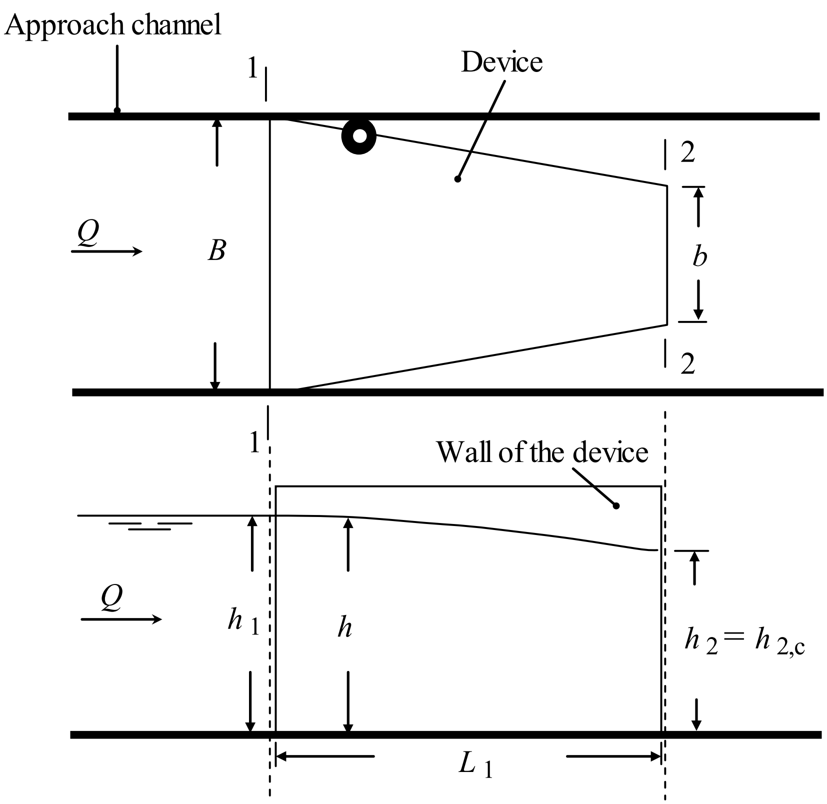


Figure 3: Longitudinal profile of the flow inside the device

As shown in the previous figures, the original Montana flume is characterized by lengths L_1 and L_2, as well as widths B and b of the inlet and outlet sections of the device. These four linear dimensions are specific to each of the original Montana flumes built.

The original Montana flumes are recognized by the narrowed width value b. For example, the 1-inch Montana flume refers to the width $b = 1$ Inch or $b = 2.45$ cm. Although Montana flume is available in twenty-two sizes, devices larger than 48 inches are rarely used in practice. Thus, in the present study, only nine of the most commonly used original Montana flumes, ranging in size from 1 inch to 36 inches, are considered.

Like the Parshall and the Venturi, the Montana flumes are so-called empirical devices on which experiments have been carried out to derive the stage-discharge relationship. Every Montana flume built is governed by its own stage-discharge relationship. Additionally, Montana flumes are not scale models of each other and are not dynamically similar, which amounts to saying, for example, that the 12-inch does not mean that its dimensions are three times that of the 4-inch.

Unlike most flow meters, manometric reading of the stage in Montana flumes is not operated in the upstream section 1-1 but in a section located inside the device (Figs. 2 and 3). The position of the manometric point gauging is predefined by the distance L_2 for each of the Montana flumes; this position must be respected to comply with the requirements for the proper use of the device. However, for the eight considered devices, calculations carried out by the authors have shown that the distance Δx, which separates the point of measurement and the inlet section 1-1 of the device, varies between 12 cm and 55 cm depending on the size of the device (Table 1). These relatively short distance Δx values should reasonably predict that the depth h_1 in the inlet section 1-1 (Fig. 3) could be assimilated to the depth h at the measurement point. However, to confirm this hypothesis, additional tests are needed. The advantage of measuring h_1 instead of h is that the position of the measuring point in the inlet section 1-1 does not change regardless of the device size, which is not currently the case for the original Montana flumes.

The characteristics of the nine Montana flumes that are considered in this study are listed in Table 1, drawn up from data available in the literature (Open channel flow, 2024).

Table 1: Characteristics of the considered original Montana flumes

Montana Flume size	B (cm)	b (cm)	Contraction Rate $\beta = b/B$	L_1 (cm)	L_2 (cm)	Δx (cm)
1-Inch	16.75	2.54	0.151	35.56	24.21	11.82
2-Inch	21.35	5.08	0.238	40.64	27.62	13.55
3-Inch	25.88	7.62	0.294	45.72	31.12	15.13
6-Inch	39.69	15.24	0.384	60.96	41.43	20.14
9-Inch	57.47	22.86	0.398	86.36	58.74	28.76
12-Inch	84.46	30.48	0.361	134.30	91.44	44.63
18-Inch	102.55	45.72	0.446	141.92	96.52	47.27
24-Inch	120.65	60.96	0.505	149.54	101.60	49.91
36-Inch	157.16	91.44	0.582	164.47	111.76	54.87

Due to the progressive decrease in the cross-sectional area inside the device, the flow gradually varies between sections 1-1 and 2-2, generating an H2-backwater curve, meaning that the depth decreases until reaching section 2-2, which the authors assume to be a control section where the flow depth $h_{2,c}$ is critical; the subscript "*c*" denotes the critical conditions. A subcritical flow in a steady state settles upstream and inside the device, particularly in section 1-1 (Fig. 3), where the flow depth is h_1. The specialized literature does not give any information on this state of the flow in section 2-2. Flow measurement in open channels is often, and even always, based on the critical state of the flow in the control section. This feature will be exploited during the investigations of the modified Montana flume.

According to the authors' extensive investigations, it would seem that there is no theoretical study carried out on the existing Montana flumes; only laboratory and field test results are available in the literature, in particular, the values of the empirical parameters of the stage-discharge relationship. This is the same as the one used in Parshall since it is expressed as follows provided that the device works with no submergence:

$$Q = K\,h^{n} \qquad (1)$$

Where Q is the flow rate, K is the flume discharge constant varying by flume size/unit, h is the depth at the point of measurement located inside the device, a few centimeters from the entrance (Table 1), and n is the discharge exponent depending upon the flume size.

There is no generalized relationship giving the flow rate Q as a function of both the contraction rate $\beta = b/B$ and the upstream depth h. For each Montana flume model, characterized by a given value of the contraction rate β, the empirical relationship of the flow rate Q was determined. There are thus as many flow rate formulas Q as there are built models, each defined by a given value of the contraction rate.

The specialized literature (Open channel flow, 2024) reveals that in regard to practical flow considerations such as approach flow as well as installation and dimensional tolerances requirements, the free-flow discharge precision of the device is likely to be closer to +/-5%, which is a common and acceptable deviation in the flow measurement field; this is valid for all models of Montana flumes built.

Table 2, which the authors have drawn up taking into account the data available in the literature (Open channel flow, 2024), lists the characteristics of the eight considered Montana flumes available in the literature, as well as the values of the coefficients K and n of the stage-discharge relationship expressed by Eq. (1) for each device. The constants n and K correspond to the depth h expressed in meters for a flow rate Q given in cubic meters per second. Table 4 also shows the significant number of measurements collected during testing on the Original Montana flumes, totalling one thousand five hundred and seventy (1570).

Table 2: The constants of the stage-discharge relationship expressed by Eq. (1) for the considered original Montana flumes

Montana Flume size	Contraction rate β	Range of the depth h (m)	Number of measurements	n	K
1-Inch	0.151	$0.0152 \le h \le 0.2134$	67	1.550	1.550
2-Inch	0.238	$0.0152 \le h \le 0.2438$	76	1.550	1.550
3-Inch	0.294	$0.0305 \le h \le 0.4572$	141	1.550	1.550
6-Inch	0.384	$0.0305 \le h \le 0.4572$	141	1.580	1.580
9-Inch	0.398	$0.0305 \le h \le 0.6096$	191	1.530	1.530
12-Inch	0.361	$0.0305 \le h \le 0.7620$	241	1.522	1.522
18-Inch	0.446	$0.0305 \le h \le 0.7620$	241	1.538	1.538
24-Inch	0.505	$0.0457 \le h \le 0.7620$	236	1.550	1.550
36-Inch	0.582	$0.0457 \le h \le 0.7620$	236	1.566	1.566

Although it allows flow measurement with tolerable accuracy, the fact remains that Eq. (1) is not rational and does not correspond to any proven principle of flow measurement. The exponent n varies, causing an unfounded change in the dimension of the coefficient K. Rational theory has shown that for a given shaped section of the device, the exponent n remains constant, regardless of the device size, depth or flow rate involved; it depends solely on the section shape of the device, and it is worth, for example, 3/2 for the rectangular section, which is the case for Montana flumes, and 5/2 for the triangular section (Bos, 1989).

The irrationality of Eq. (1) is among the many disadvantages of the Montana flume in its original form. These constraints will be largely reduced, and their effects will be mitigated by advocating and investigating the modified Montana flume, which will be described in the appropriate section.

Dimensional analysis and discharge coefficient dependency

Dimensional analysis is an important tool that helps uncover the parameters that influence a given flow parameter. In what follows, the dimensionless parameters that influence the discharge coefficient of the device, particularly the relative upstream depth h/b or h/B, are highlighted.

Regarding Montana flume described previously, the nine geometric and hydraulic parameters involved can be listed as follows:

The flow rate Q passing through the cross-section of width B, the flow depth h measured inside the device, the inlet device width B, the outlet device width b, the acceleration g due to gravity, the density ρ of the flowing liquid, the surface tension σ, the dynamic viscosity μ of the flowing liquid, and the horizontal length L_1 of the contraction device.

Let us denote by f the functional that connects these nine parameters. On may write the following:

$$f\left(Q,\rho,g,h,B,b,L_1,\mu,\sigma\right)=0 \quad (2)$$

Using the Vashy-Buckingham π theorem (Langhaar, 1962), the flow rate Q can be written as the following function of dimensionless parameters:

$$Q=g^{1/2}h^{3/2}\varphi\left(\frac{\rho g h^{3/2}}{\mu},\frac{\rho g h^2}{\sigma},\frac{h}{L_1},\frac{h}{B},\frac{h}{b}\right) \quad (3)$$

The symbol φ is the functional that translates the discharge coefficient C_d relationship.

One may recognize that the first two dimensionless parameters inside the parentheses of Eq. (3) are the Reynolds number R and the Weber number W, respectively. Hence, the discharge coefficient C_d can be expressed as follows:

$$C_d=\varphi\left(R,W,\frac{h}{L_1},\frac{h}{B},\frac{h}{b}\right) \quad (4)$$

It is worth noting that the effect of the Reynolds number R on the discharge coefficient C_d is not significant, even negligible, due to the turbulent nature of the flow. The Weber number W, which translates the effect of the surface tension σ, only has an influence in the case of small flow rates Q or for devices of reduced dimensions. On the other hand, the influence of the length L_1 can be neglected provided that the ratio L_1/h exceeds the threshold value defined by the standardized dimensions authoritative publications, as ASTM D1941-91 (Open channel flow, 2024).

Taking all these considerations into account, Eq. (4) reduces to:

$$C_d=\psi\left(\frac{h}{B},\frac{h}{b}\right) \quad (5)$$

However, the ratio h/b can be written as follows:

$$\frac{h}{b}=\frac{h/B}{b/B}=\frac{h}{B}\beta^{-1} \quad (6)$$

As a result, Eq. (5) takes the following final form:

$$C_d=\zeta\left(\beta,\frac{h}{B}\right) \quad (7)$$

Thus, Eq. (7) reveals that the discharge coefficient C_d depends on both the contraction rate β of the device and the relative upstream flow depth h/B. The ζ functional relationship will be clearly defined in this study through the use of both theoretical and experimental data.

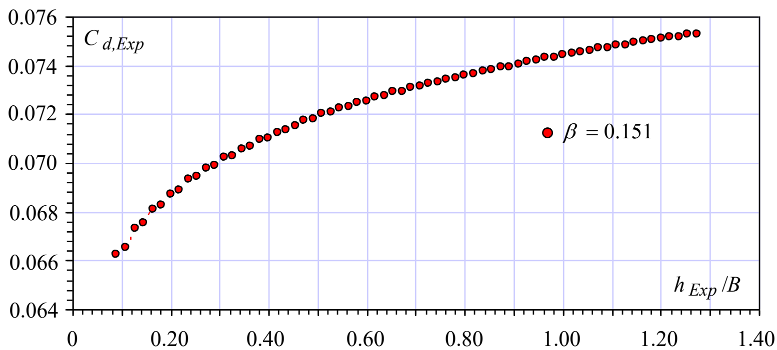
In the meantime, it would be informative to examine the degree to which the upstream flow depth h/B influences the discharge coefficient C_d and hence the discharge Q from a qualitative point of view. This will be shown in the next paragraph.

Qualitative influence of the upstream relative flow depth on the discharge coefficient

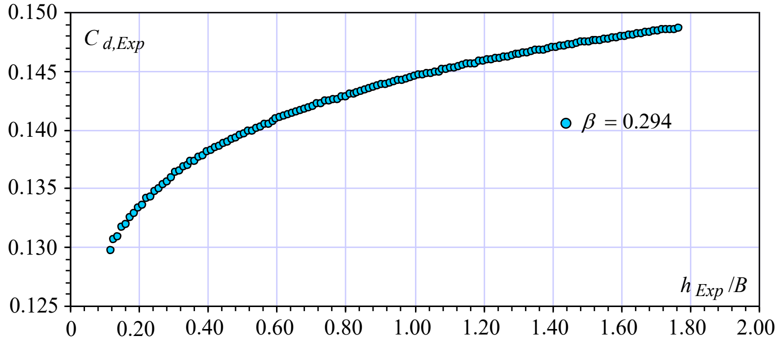
Based on the analysis of one thousand five hundred and seventy (1570) experimental measurements of the couple (Q_{Exp}; h_{Exp}) available in the specialized literature (Open channel flow, 2024), involving the nine Montana flumes that are the subject of the present study, Fig. 4 shows the experimental variation in the experimental discharge coefficient $C_{d,Exp}$ as a function of the experimental upstream relative depth h_{Exp}/B for the tested corresponding contraction rate β. The discharge coefficient $C_{d,Exp}$ was calculated according to the following relationship:

$$C_{d,Exp} = Q_{Exp}/(\sqrt{2g}\ B\ h^{3/2})$$

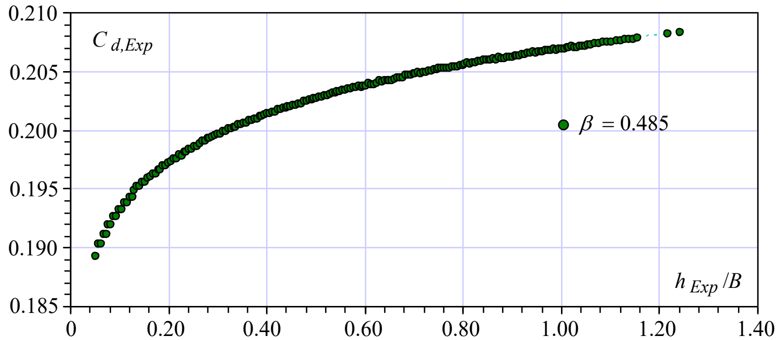
To avoid a clutter of the curves and to better visualize the influence of h/B on C_d, the variation in $C_{d,Exp}$ against h_{Exp}/B has not been represented in the same figure for all the Montana flumes. However, the few cases in Fig. 4 are representative of all the others, whose $C_{d,Exp}(h_{Exp}/B)$ variation is similar.



a)



b)



c)

Figure 4: Influence of the upstream depth *h/B* on the discharge coefficient C_d for some original Montana flumes. a) β = 0.151 (1-inch); b) β = 0.294 (3-inch); c) β = 0.485 (9-inch)

Fig. 4 first shows that the relative upstream depth *h/B* undoubtedly influences the discharge coefficient C_d since the variation in $C_{d,Exp}(h_{Exp}/B)$ is represented by power type curves. The dimensional analysis predicted this result. If the influence of h/B did not exist, then the variation in C_d as a function of h/B would be represented by horizontal straight lines, which means that the discharge coefficient C_d would be a constant depending solely on the considered contraction rate β, i.e., $C_d = f(\beta)$.

It is useful to emphasize that these results are the opposite of those found during the observations carried out on sharp-edged width constriction. For this device, no influence of the upstream depth on the discharge coefficient was observed for small contraction rates such that $\beta \leq 0.45$ (Achour and Amara, 2021a), while other observations have been able to show an influence of the upstream flow depth on the discharge coefficient for large values of β such as β = 0.50; 0.60; 0.70; and 0.80 (Hager, 1988; Goel, 2015).

MODIFIED MONTANA FLUME

Description of the device

Instead of constructing the device in one piece, as shown in Fig. 1, it would be more attractive to consider the removable elements shown in Fig. 5. When these elements are arranged in a rectangular channel, they create a converging passage of rectangular cross-sections along the entire length L_1, similar to that of Fig. 1. It is worth noting that such a converging passage shape can be created using width contraction, functioning like two shutters of a door of reduced thickness (Goel et al., 2015). However, this device configuration has several disadvantages such as high instability, especially at high flow rates, major water leaks, and difficulty in fastening to the side walls of the approach channel. In contrast, the triangular prismatic element of the configuration depicted in Figs. 5 and 6, provides better stability through its base, and also in particular when filled with a material such as sand to weigh it down, thus preventing the element from being carried away by the water current. Fastening such an element to the approach channel side walls is not required.

Fig. 6 shows the details of one of the two triangular prismatic removable elements embodying the device.

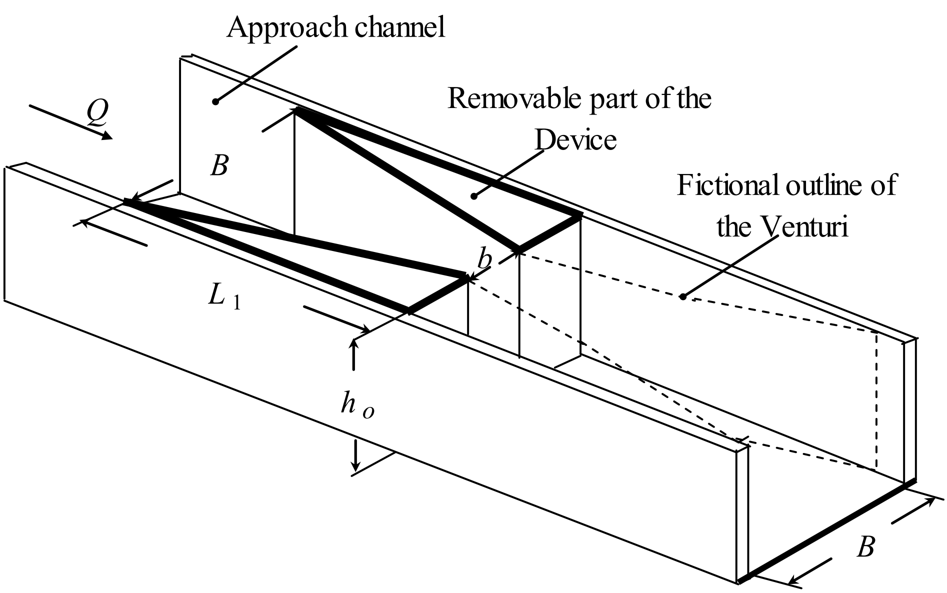


Figure 5: Perspective view sketch of the modified Montana flume inserted into a rectangular approach channel

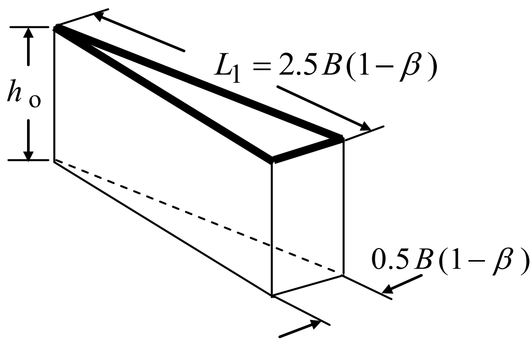


Figure 6: Perspective view of one of the two elements making up the modified Montana flume

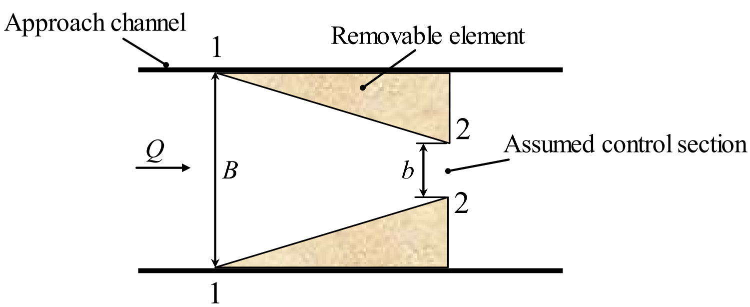


Figure 7: Plan view of the modified Montana flume showing the two removable triangular prismatic elements

As in the case of the original Montana flume, the initial rectangular cross-section of the new device is 1-1 of width B equal to that of the original Montana flume, while the exit terminal rectangular section is 2-2 of width b different from that adopted for the original Montana flume, as will be subsequently demonstrated (Fig. 7). These two linear dimensions form the contraction rate $\beta = b/B$ of the device such that $0 < \beta < 1$. The device elements have a height h_o equal to the approach channel's (Fig. 5).

Sizing the element represented in Fig. 6 consists of choosing the appropriate length L_1 and the central opening width b. To do so, the designer must select an arbitrary value of the contraction rate β among a range of predefined optimal values derived from the analysis of the experimental data collected on the original Montana flume. Once the contraction rate β is fixed, the width b is simply computed as $b = B\beta$, where the approach channel width B value is given corresponding also to the width of the inlet cross-section of the device. The values of the width b of the modified Montana flumes were calculated and reported in Table 3, depending on the size of the device. On the other hand, the procedure for the calculation of the optimal β values will be presented later.

In the paragraph dedicated to this purpose, it will be shown that the contraction rate of the modified Montana flume represented in Fig. 5 is greater than that of the original Montana flume for the same value of inlet cross-section width B. This means that the width b is greater, thus implying a lesser width $(B - b)/2$ of the triangular prismatic element shown in Fig. 6. In addition, this finding will result in a significant reduction in the length L_1 of the elements designing the modified Montana flume (Fig. 6) compared to that of the original Montana flume.

To avoid any flow separation from the wall of the device that could occur under certain hydraulic upstream flow conditions, and based on in-depth calculations, the authors recommend calculating the length L_1 of the triangular prismatic element of the modified Montana flume (Fig. 6) according to the following relationship:

$$L_1 = \frac{5}{2} B(1-\beta) \tag{8}$$

In Eq. (8), β must be chosen among the predefined optimal contraction rate values, determination procedure will be clarified in one of the next sections.

In the schematic facility depicted in Fig. 5, it is important to recognize the obvious advantage of the design and the construction of the device compared to Venturi represented fictitiously by the dashed line. The modified Montana flume requires a minimum of space, practically one-third of that occupied by the Venturi formed by three static parts, specifically an inlet converging section, a throat, and an outlet discharge sections. Furthermore, it will be demonstrated that the modified Montana flume is much more economical than the original Montana flume due to its reduced dimensions.

However, it is worth pointing out that the simplest and most attractive from an economic point of view measuring flow devices in open channels are those corresponding to the length L_1 reduced to 1 or 2 centimeters, corresponding to the thickness of a thin plate. In this case, the device is reduced to a sharp-edged width constriction formed by two thin plates, which has recently been the subject of relevant theoretical and experimental investigations (Achour and Amara, 2021e).

Theoretical discharge and discharge coefficient relationships

As depicted in Fig. 7, it is assumed that section 2-2 is a control section where the flow depth is critical. Accurately, one does not know exactly which cross-section is the location of the critical depth, perhaps immediately upstream of the cross-section 2-2. Assume herein, for the convenience of the theoretical development, that cross-section 2-2 corresponds to the control section. Deviations in the flow rate calculation that could be caused by such an assumption, in the event that the criticality of the flow is not ensured in the section 2-2, will be corrected by the effects of a correction factor derived from the experimental results available in the literature (Open channel flow, 2024).

It is useful to note that the upstream flow depth measurement location, noted Δx in Table 1, is precisely indicated for each of the original Montana flume models, which must be respected. If one adopts the configuration of the device represented in Fig. 7, the location of the depth measurement corresponds to section 1-1 immediately upstream or at the inlet of the device, regardless of its size. The depths h_1 and h indicated in Fig. 3 are not exactly the same due to the decrease in the flow depth in the device according to an H2 backwater curve type.

As Montana flume, the considered device works under free-spilling flow off the end of the flume, i.e., at the location of section 2-2.

The critical depth in rectangular cross-section 1-1 (Fig. 7) is written as:

$$h_{1,c} = \left(\frac{Q^2}{gB^2} \right)^{1/3} \tag{9}$$

Where the subscript « c » denotes the critical flow conditions.

On the other hand, the critical depth in rectangular cross-section 2-2 is as follows:

$$h_{2,c} = \left(\frac{Q^2}{gb^2} \right)^{1/3} \tag{10}$$

The ratio of Eq. (9) to Eq. (10) gives:

$$\frac{h_{1,c}}{h_{2,c}} = \left(\frac{b}{B} \right)^{2/3} = \beta^{2/3} \tag{11}$$

Thus:

$$h_{2,c} = h_{1,c}\,\beta^{-2/3} \tag{12}$$

The convergence caused by the walls of the device leads to the establishment of a critical flow regime in section 2-2 or in a section immediately upstream. Assume that there is no head loss between sections 1-1 and 2-2 due to the short distance between them, meaning that $H_1 = H_2$, where H_1 and H_2 are the total heads in sections 1-1 and 2-2, respectively. Considering the equality of the total heads between sections 1-1 and 2-2 and expressing the condition of criticality in section 2-2, one may write the following:

$$H_1 = H_2 = \frac{3}{2} h_{2,c} \tag{13}$$

Combining Eqs. (12) and (13) results in the following:

$$H_1 = \frac{3}{2} h_{1,c}\,\beta^{-2/3} \tag{14}$$

Hence:

$$\frac{H_1}{h_{1,c}} = \frac{3}{2}\beta^{-2/3} \tag{15}$$

Considering the effect of the approach flow velocity, the total head H_1 is written as follows:

$$H_1 = h_1 + \frac{Q^2}{2gB^2 h_1^2} \tag{16}$$

Dividing both sides of Eq. (16) by $h_{1,c}$ yields the following:

$$\frac{H_1}{h_{1,c}} = \frac{h_1}{h_{1,c}} + \frac{Q^2}{2gB^2 h_{1,c} h_1^2} \tag{17}$$

Inserting Eq. (9) into Eq. (17) and rearranging allows us to write the following:

$$\frac{H_1}{h_{1,c}} = \frac{h_1}{h_{1,c}} + \frac{1}{2(h_1 / h_{1,c})^2} \tag{18}$$

Considering Eq. (15), Eq. (18) can be rewritten as follows:

$$\frac{H_1}{h_{1,c}} = \frac{h_1}{h_{1,c}} + \frac{1}{2(h_1 / h_{1,c})^2} = \frac{3}{2}\beta^{-2/3} \tag{19}$$

Let us adopt the following dimensionless parameter:

$$\frac{h_1}{h_{1,c}} = h_1^* \tag{20}$$

Inasmuch as the upstream flow depth h_1 is greater than the critical depth $h_{1,c}$ in section 1-1 due to the subcritical nature of the approaching flow, then h_1^* is greater than unity, i.e., $h_1^* > 1$.

Inserting Eq. (20) into Eq. (19) results in the following:

$$h_1^* + \frac{1}{2 h_1^{*2}} = \frac{3}{2}\beta^{-2/3} \tag{21}$$

Eq. (21) shows that the dimensionless parameter h_1^* depends solely on the contraction rate β. By expanding Eq. (21), one may obtain the following third-degree equation in h_1^*

$$h_1^{*3} - \frac{3}{2}\beta^{-2/3} h_1^{*2} + \frac{1}{2} = 0 \tag{22}$$

Eq. (22) admits three real roots, of which only one satisfies the condition $h_1^* > 1$. This was determined by the method described by Spiegel (1974), and the final result is as follows:

$$h_1^* = \beta^{-2/3}\left\{ \cos\left(\frac{1}{3}\cos^{-1}\left[1 - 2\beta^2\right]\right) + \frac{1}{2} \right\} \tag{23}$$

However, combining Eqs. (9) and (20) results in the following:

$$Q = \frac{1}{\sqrt{2}}\sqrt{2g}\, B \frac{h_1^{3/2}}{h_1^{*3/2}} \tag{24}$$

Writing Eq. (24) in the form of the stage-discharge relationship generally governing weirs and flow meters, one may obtain the following:

$$Q = C_d \sqrt{2g}\, B\, h_1^{3/2} \tag{25}$$

Where C_d is the discharge coefficient of the device. Considering Eqs. (23), (24), and (25), the discharge coefficient C_d is expressed as follows:

$$C_d = \frac{1}{\sqrt{2}}\beta\left\{ \cos\left(\frac{1}{3}\cos^{-1}\left[1 - 2\beta^2\right]\right) + \frac{1}{2} \right\}^{-3/2} \tag{26}$$

Eq. (26) reveals that the theoretical discharge coefficient C_d depends exclusively on the contraction rate β. However, as predicted by the dimensional analysis, the discharge coefficient C_d should also depend on the relative upstream flow depth h/B. It is worth noting that there is no theory available in the literature that could model the effect of h/B on C_d. Only the analysis of observations could achieve this goal when considering this effect. Inserting Eq. (26) into Eq. (25), the following theoretical stage-discharge relationship of the modified Montana flume is obtained:

$$Q = \frac{\sqrt{2}}{2}\beta\left\{ \cos\left(\frac{1}{3}\cos^{-1}\left[1 - 2\beta^2\right]\right) + \frac{1}{2} \right\}^{-3/2} \sqrt{2g}\, B\, h_1^{3/2} \tag{27}$$

However, Eq. (27) is only a theoretical relationship, requiring subjected to verification tests using experimental data available in the literature, especially those carried out during the original Montana flumes experiment involving the upstream depths h reported in Table 2. The corresponding ranges of the involved flow rates Q are available in the specialized literature (Open channel flow, 2024), which reports flow rates Q varying within the following wide range: 0.0921 l/s $\leq Q \leq$ 1427 l/s.

Optimal contraction rate of the modified Montana flume

For each original Montana flume reported in Table 2, a modified Montana flume is defined with the same inlet width B but with a different outlet width b, implying a different contraction rate β value.

The suitable estimation approach of the contraction rate optimal values of the modified Montana flumes is as follows. One considers the experimental data collected on each original Montana flume reported in Table 2, knowing the range of flow rates Q corresponding to that of depths h indicated in said Table. Afterwards, for each original Montana flume, the optimal value of the contraction rate β_{opt} of the corresponding modified Montana flume was determined by optimizing the theoretical stage-discharge relationship (27), with the aid of inherent mathematical tools. This amounts to determining, for the series of experimental values of Q and h, the optimal value of β, characterizing the modified Montana flume, which would minimize the deviations between the experimental flow rates and those calculated by the theoretical relationship (27) for the given value of the width B. The nine Montana flumes reported in Table 2 have been subjected to the same approach.

The previously described optimization procedure rightly accounts for the assumed simplifying assumptions, such as the approximation made when stating the equality of the depths h_1 and h, or $H_1 = H_2$.

The reliability of the optimization procedure is estimated by comparing, at the end of the process, the experimental flow rates and those calculated according to theoretical Eq. (27), for each of the Montana flumes reported in Table 2. The results obtained from the previously described optimization procedure, especially the resulting values of $\beta_{opt.}$, are grouped in Table 3.

Table 3: Original and modified Montana flumes parameters. Optimal contraction rates $\beta_{opt.}$ computed according to an optimization procedure

Original Montana Flumes				Modified Montana Flumes		
Size	β	***b*** **(cm)**	L_1 **(cm)**	$\beta_{opt.}$	***b*** **(cm)**	L_1 **(cm) Eq. (8)**
1-Inch	0.151	2.54	35.56	0.1817517	3.044	34.26
2-Inch	0.238	5.08	40.64	0.28314724	6.045	38.26
3-Inch	0.294	7.62	45.72	0.34973187	9.051	42.07
6-Inch	0.384	15.24	60.96	0.44936404	17.84	54.64
9-Inch	0.398	22.86	86.36	0.48522771	27.88	73.96
12-Inch	0.361	30.48	134.30	0.43908722	37.08	118.44
18-Inch	0.446	45.72	141.92	0.52409018	53.74	122.01
24-Inch	0.505	60.96	149.54	0.58381812	70.44	125.53
36-Inch	0.582	91.44	164.47	0.64915714	102.02	137.85

Table 3 indicates that the contraction rates of the modified Montana flumes are higher than those of the original Montana flumes, knowing that the width B, reported in Table 1, has been kept the same. This has the consequence of increasing the width b of the outlet section of the device, reducing the width of the triangular prismatic element (Fig. 6) to its minimum possible, constituting an economic gain. In addition, Table 3 highlights that the length L_1 of modified Montana flumes, computed using Eq. (8) for $\beta = \beta_{opt.}$ are lower than those of the original Montana flumes, especially for large $\beta_{opt.}$ values, also constituting an economic gain. Moreover, to better underline the prominent economic gain inciting the use of the advocated device, the calculation showed that the modified Montana flumes are more economical since they require a smaller material surface area than the original Montana flumes regardless of the size. To illustrate this statement, the authors were able to show, for instance, that the original 36-inch Montana flume, of h_o = 1 m height, requires for its manufacture a material surface area of more than 4.6 m^2, whilst the corresponding modified Montana flume develops a flattened surface area less than 3.85 m^2 only, for the same height. In this case, compared to the modified Montana flume, the fabrication of the corresponding original 36-inch Montana flume requires 16.3% more material.

On the other hand, Table 3 safely assists in sizing the modified Montana flume, more precisely the triangular prismatic element that make up the device in accordance with Fig. 6. Indeed, Table 3 provides the values of the following required parameters $\beta_{opt.}$ and L_1, bearing in mind that both the width B and the channel height ho are given. It is useful to remember that the width B of the approach channel is also equal to the inlet width of both the original and modified Montana flumes.

It is useful to recall that the optimal values of the contraction rateβ, reported in Table 3, were determined for modified Montana flumes intending to measure flow rates up to one thousand four hundred and twenty-seven litres per second (1427 l/s). Thus, Table 3 encompasses an exhaustive practical wide range of configurations under various upstream flow conditions that may involve low or high flow rates.

Note that if one prefers to design the modified Montana flume in the common form of the original Montana flume as depicted in Fig. 1, then the designer has safely to adopt the values of the outlet width b and length L_1 reported in Table 3, for the corresponding value of the inlet width B. However, as previously highlighted, this design is less economical than that advocated by the authors because its fabrication requires much more material. Therefore, the authors do not recommend such a design.

Corrected theoretical discharge coefficient relationship

Although the contraction rates of the modified Montana flumes are optimal, a deviation, which could reach 10% in no more than two cases, was observed between the theoretical discharge coefficients calculated according to Eq. (26) for $\beta = \beta_{opt}$, and the experimental discharge coefficients calculated according to the following relationship:

$$C_{d,Exp} = \frac{Q_{Exp}}{\sqrt{2g}\, B\, h_{Exp}^{3/2}} \tag{28}$$

The subscript "*Exp*" denotes "Experimental".

Eq. (28) involved all the upstream depths reported in Table 2 as well as the corresponding flow rates, i.e., 1.52 cm ≤ h ≤ 76.20 cm and 0.0921 l/s ≤ Q_{Exp} ≤ 1427 l/s.

A thorough examination of the problem, involving the 1570 experimental values of Q and h, revealed that the theoretical discharge coefficient of the modified Montana flume governed by Eq. (26) for $\beta = \beta_{opt}$ is influenced by the relative flow depth h/B, or h_1/B, as predicted by the dimensional analysis. Thus, statistical analysis afforded that the most appropriate relationship between the experimental discharge coefficient and the corrected theoretical discharge coefficient is expressed as follows:

$$C_{d,Exp} = 0.9523\, C_{d,Th}\, \beta^{-0.0607} \left[\left(\frac{h_1}{B} \right)^{0.2496} + \beta^{1.5858} \right]^{0.243} \tag{29}$$

Let us recall that in Eq. (29), $C_{d,Th}$ is the theoretical discharge coefficient of the modified Montana flume, given by Eq. (26) for $\beta = \beta_{opt}$. In Eqs. (29), $C_{d,Exp}$ should be considered as the modified Montana flume corrected discharge coefficient. Hence, Eq. (29) can be rewritten as follows:

$$\frac{C_{d,Exp}}{C_{d,Th}} = 0.9523\, \beta^{-0.0607} \left[\left(\frac{h_1}{B} \right)^{0.2496} + \beta^{1.5858} \right]^{0.243} \tag{30}$$

Table 4 highlights the results of the comparison between the values of the modified Montana flume discharges computed using Eq. (25) where the discharge coefficient is given by Eq. (29) along with Eq (26) for $\beta = \beta_{opt}$ and the original Montana flumes discharges given by Eq. (1) along with Table 2.

Table 4: Deviation between MMFs corrected theoretical discharges and the OMFs discharges. OMF = Original Montana Flume; MMF = Modified Montana Flume

OMF Size/MMF-β_{opt}	Deviation $\Delta Q/Q$ (%) between OMF and MMF			Ranges of relative upstream depths h_1/B
	Maximum	Minimum	Average	
1-Inch/0.181	1.790	0.0988	1.251	[0.0907; 1.274]
2-Inch/0.283	1.094	0.320	0.570	[0.0712; 1.142]
3-Inch/0.349	1.878	1.413	1.512	[0.118; 1.767]
6-Inch/0.449	4.969	0.0353	2.857	[0.0768; 1.152]
9-Inch/0.485	2.078	0.000	1.038	[0.053; 1.242]
12-Inch/0.439	4.977	0.0078	1.318	[0.0361; 0.9022]
18-Inch/0.524	1.761	1.180	1.474	[0.0297; 0.743]
24-Inch/0.584	2.189	0.0094	1.409	[0.0378; 0.6316]
36-Inch/0.649	4.970	0.0323	2.947	[0.029; 0.4848]

As indicated in Table 4, the use of modified Montana flumes results in discharge deviations less than or equal, in a few cases, to the deviation of approximately 5% caused by the original Montana flumes.

The meaningful observation that must be pointed out is that Eqs. (25), (26) and (29) together represents the unique relationship allowing calculating the flow rate sought using the modified Montana flume whatever its size. This procedure advantageously replaces the irrational Eq. (1) which must be applied according to the size of the original Montana flume and the corresponding values of the coefficients n and K.

CONCLUSION

The Montana flume, one of the most economical devices for measuring flow in open channels and available in twenty-two sizes, has been revisited from the theoretical and design points of view to make it an even simpler and more economical device governed by a rational stage-discharge relationship.

According to the admitted classification, the study was limited to nine practical models ranging in size from 1 inch to 36 inches. The original Montana flumes considered herein are the empirical devices, similar to many other flumes, which have been the subject of an intense experimental program involving flow rates Q varying between 0.0921 l /s and 1427 l /s, corresponding to upstream flow depths h, ranging between 1.52 cm and 76.20 cm. These wide ranges involve practically all configurations, both in the laboratory and in the field.

As with many flumes, the empirical stage-discharge relationship derived from laboratory and field tests has been expressed for each device as follows: $Q = K\,h^n$, where Q is the flow rate, K is the flume discharge constant, h is the measured upstream depth, and n is the discharge exponent. The major drawback of the previous relationship resides in the fact that n varies according to the size of the original device. For the nine original Montana flumes considered in the study, the exponent n varies between 1.550 and 1.566. The variation of n causes an inappropriate change in the dimensions of the coefficient K, which does not correspond to any proven principle. Since the cross-section of the device is rectangular in shape, it has been shown by numerous studies that the exponent n should be equal to 1.50, regardless of the width of the section. If the tests reveal that the exponent n deviates significantly from this value, there is reason to suspect effects of the upstream relative flow depth on the stage-discharge relationship, in particular the discharge coefficient C_d.

To give more credit and rationality to the stage-discharge relationship governing the device, the authors intended to derive it analytically. By suitably manipulating the equation of the energy applied between the inlet and outlet sections of the device, the discharge coefficient relationship was deduced by solving a third degree equation, involving only the contraction rate β of the device, i.e., $C_d = f(\beta)$. As a result, the stage-discharge relationship was successfully derived, rightly showing that $Q \propto h^{3/2}$ in accordance with known principles, and this result is valid for all the considered devices regardless of their size.

Considering the wide range of experimental values of the couple (Q; h) available in the literature, totalling one thousand five hundred and seventy (1570), the theoretical stage-discharge relationship was subjected to an optimization process performed by inherent mathematical tools. This optimization process allowed calculating the optimal contraction rate β_{opt} for each of the considered modified Montana flumes, which gives the device geometry such that the deviations between the experimental and theoretical flow rates are the minimum possible. The optimum contraction rates of the modified Montana flumes are higher than those of the original Montana flumes, resulting in smaller outlet widths than those of the original Montana flumes, earning the name "Modified Montana flume" for the new device.

The last step of the study consisted of correcting the theoretical discharge coefficient relationship $C_d = f(\beta_{opt})$ for the effects of the relative upstream flow depth h_1/B, as predicted by the dimensional analysis. The stage-discharge relationship thus corrected causes deviations less than or equal to those induced from using the governing stage-discharge relationship of the original Montana flumes. The largest deviation of approximately 5% was observed only in a few cases.

Finally, the suitable dimensions of the modified Montana flume have been recommended giving the device the most economical geometry compared to the original Montana flumes.

It is worth noting that in the field of flow measurement, what practitioners expect from designers, hunger for and even strongly recommend, is to design attractive devices that allow calculation. In addition, the relationships that govern both the discharge and the discharge coefficient must be based on proven fundamental principles of hydraulics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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