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CRITICAL FLOW IN A TRIANGULAR-SHAPED CHANNEL

ECOULEMENT CRITIQUE DANS UN CANAL DE FORME TRIANGULAIRE

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ABSTRACT

The functional $\psi(y_c;m;S_0;\varepsilon;\nu) = 0$ relationship has been well defined and theoretically established for the triangular-shaped channel, where y_c is the critical depth, m is the side slope, S_0 is the channel bottom slope, ε is the absolute roughness, and ν is the kinematic viscosity of the flowing water. A thorough investigation of the function revealed that the critical depth y_c is governed by a cubic equation without second order term. Its analytical resolution is very easy when one uses the circular or hyperbolic trigonometry.

The article ends with the study of the special case of the smooth triangular-shaped channel of a 90° apex angle by examining the equation that governs the critical depth It turned out that y_c is given by an explicit equation, as a function of m, S_0, and ν. In addition, it has been demonstrated that, for such a canal, the more the slope S_0 increases, the more the critical depth decreases. Moreover, it was observed that, for the same slope S_0, the critical depth decreases as the side slope m increases, i.e. when the apex angle of the channel increases. For slopes S_0 less than 0.0012, the critical depths are so high that they are outside the practical context. As a matter of fact, for the slope $S_0 = 0.0012$, the critical depth already reaches more than 5m.

Keyword: Triangular channel, critical flow, normal depth, critical depth, discharge, slope.

RESUME

La relation fonctionnelle $\psi(y_c;m;S_0;\varepsilon;\nu) = 0$ a été bien définie et théoriquement établie pour le canal de forme triangulaire, où y_c est la profondeur critique, m est la pente des parois latérales du canal, S_0 est la pente du fond du canal, ε est la rugosité absolue et ν est la viscosité cinématique de l'eau en écoulement. Une étude approfondie de la fonction a révélé que la profondeur critique y_c est régie par une équation de troisième degré sans terme du second ordre. Sa résolution analytique est très aisée lorsqu'on utilise la trigonométrie circulaire ou hyperbolique.

L'article se termine par l'étude du cas particulier du canal lisse de forme triangulaire de 90° d'angle d'ouverture en examinant l'équation qui régit la profondeur critique. Il s'est avéré que y_c est donnée par une équation explicite, en fonction de m, S_0 et ν.En outre, il a été démontré que, pour un tel canal, plus la pente S_0 augmente et plus la profondeur critique diminue. Il a été également observé que, pour une même pente S_0, la profondeur critique diminue au fur et à mesure que la pente latérale m augmente, c'est-à-dire lorsque l'angle au sommet du canal augmente. Pour les pentes S_0 inférieures à 0,0012, les profondeurs critiques sont si élevées qu'elles sortent du contexte pratique. En effet, pour la pente S_0 = 0,0012, la profondeur critique atteint déjà plus de 5 m.

Mots clés : Canal triangulaire, profondeur critique, profondeur normale, débit, pente.

INTRODUCTION

The triangular-shaped canal is a special case of the trapezoidal-shaped channel with a bottom width b equal to zero. Thus, the relations which govern the flow in the triangular channel can then be deduced from those of the trapezoidal channel by writing $b = 0$. Unlike other geometric shapes of channels, it is not possible for the triangular-shaped channel to have a relative roughness, such as ε/b for the rectangular and trapezoidal channels or ε/D for the circular conduit, where ε is the absolute roughness, and D is the diameter of a circular conduit. These ratios are constant for a given channel or conduit. For the triangular-shaped channel, it is possible to form the ratio ε/y, where y is the flow depth. However, this ratio varies according to y, which is not convenient for calculations. It is only constant for a given depth of the flow.

The most economical triangular-shaped channel, characterized by a minimum of wetted perimeter P, corresponds to the triangular shape with an apex angle of 90° (Chaudhry, 2008).

In practice, the triangular-shaped channel can be found in small hydraulic drainage systems. But its most widespread application concerns man-made lined irrigation ditches, raised up using pillars, that artificially supplies water to an area of dry land, or for the irrigation needs of a given agricultural plot. Depending on the upstream flow conditions, the ditch may be the seat of a supercritical flow of shallow depth, which does not then allow siphoning of water and evacuate it outside the ditch. To remedy this situation, a hydraulic jump is created artificially by the installation of a sill at the upstream of a well-defined height in order to raise the level of water and proceed to siphoning, either manually or automatically when the depth of the flow is sufficient to prime the siphon. The appearance of the hydraulic jump is obviously favoured by the sill but also by the supercritical flow regime associated with weak slopes of the ditch. These considerations have sparked the interest of many research workers to investigate both problems of normal depth and the hydraulic jump in the triangular-shaped channel (Hager and Wanoschek, 1985; Achour and Debabèche, 2003; Swamee and Rathie, 2004).

Uniform flow is characterized by a particular depth called the critical depth. It is the depth which corresponds to a minimum specific energy for a given discharge, or to a maximum discharge for a given specific energy (Chow, 1959). Until now, the critical depth has been deduced from the criticality criterion which results in a Froude number equal to unity, whatever the shape of the channel or the conduit (Swamee, 1993; Vatankhah and Easa, 2011; Chow, 1959; Henderson, 1966: French, 1985). But this criterion can give real critical depths as it can lead to fictitious critical depths. Although it is particular, the critical depth is a normal depth which must therefore depend on the characteristics of the flow and the geometry of the channel, such as the linear dimensions of the canal, the slope of the channel, the absolute roughness, and the kinematic viscosity of the flowing water. During our investigations, it was found that the literature did not provide any document, report or research article on this subject matter. This knowledge gap on this subject does not only concern the triangular-shaped channel but also all known forms of canals.

It was not until the year 2020 that the authors proposed the study of the critical flow in a circular conduit, taking into account the effects of all flow and conduit parameters (Achour and Amara, 2020).

The present study is a continuation of the authors' investigations of critical flow in open channels. It is interested in the triangular-shaped channel by attempting to examine the possibility of theoretically establishing the general relationship that governs the critical depth in this type of channel.

GEOMETRIC PROPERTIES

Figure.1 is a schematic representation of the definition of the studied triangular-shaped canal and the various parameters associated with it and adopted in this study.

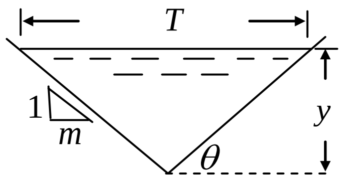

Figure 1: Schematic representation of the studied triangular-shaped channel

The geometric properties are the water area *A*, the top width *T* at the water surface, the wetted perimeter *P*, and the hydraulic radius R_h. These can be expressed for a triangular-shaped channel respectively as:

$$A = my^2 \tag{1}$$

$$T = 2my \tag{2}$$

$$P = 2y\sqrt{1+m^2} \tag{3}$$

$$R_h = \frac{my}{2\sqrt{1+m^2}} \tag{4}$$

Where, *m* is the side slope of the channel computed as horizontal distance divided by vertical distance i.e. $m = \text{cotg}\,\theta$ where θ is the angle formed by the side wall of the channel with respect to the horizontal, and *y* is the flow depth measured perpendicular to the bottom of the channel. The most economical triangular channel, corresponding to the minimum wetted perimeter, is obtained for $m = 1$ or $\theta = 45$ °, i.e. to an apex angle of 90°. According to Eqs. (3) and (4), the wetted perimeter and the hydraulic radius would become respectively as:

$$P = 2\sqrt{2}y \tag{5}$$

$$R_h = \frac{y}{2\sqrt{2}} \tag{6}$$

SIMULTANEOUS VARIATION OF CRITICAL AND NORMAL DEPTHS

In an earlier study, Achour and Amara (2020) established the relationship between the characteristics of critical and normal flows, applicable to all shapes of channel and conduit. This is expressed as:

$$\frac{A_c^{3/2}}{\sqrt{T_c}} = -4\sqrt{2}\,\frac{A_n^{3/2}}{P_n^{1/2}}\sqrt{S_0}\,\log\left(\frac{\varepsilon}{14.8R_{h,n}} + \frac{10.04}{R^*}\right) \tag{7}$$

Where ε is the absolute roughness, and R^* is the shear Reynolds number, expressed as:

$$R^* = 32\sqrt{2}\,\frac{\sqrt{gR_{h,n}^3 S_0}}{\nu} \tag{8}$$

ν is the kinematic viscosity of the flowing liquid. The subscripts "*c*" and "*n*" denote the critical and the normal flows respectively.

Taking into account Eqs.(1) to (4), Eq.(7) becomes:

$$\frac{m^{3/2} y_c^3}{\sqrt{2my_c}} = -4\sqrt{2}\,\frac{m^{3/2} y_n^3}{\left(2y_n\sqrt{1+m^2}\right)^{1/2}}\sqrt{S_0}\log\left(\frac{\varepsilon}{7.4my_n}(\sqrt{1+m^2}) + \frac{10.04}{R^*}\right) \tag{9}$$

After some simplifications and rearrangements, Eq.(9) can be rewritten as follows:

$$y_c^{5/2} = -4\sqrt{2}\,\frac{y_n^{5/2}}{\left(1+m^{-2}\right)^{1/4}}\sqrt{S_0}\log\left[\frac{\varepsilon}{7.4y_n}(\sqrt{1+m^{-2}}) + \frac{10.04}{R^*}\right] \tag{10}$$

On the other hand, Eq.(8) becomes:

$$R^* = 16\,\frac{\sqrt{g\,y_n^3 S_0}}{\nu}(1+m^{-2})^{-3/4} \tag{11}$$

Eq.(10), along with Eq.(11), is the general relationship which links the characteristics of the critical and normal flows in a triangular-shaped channel. It is a function of six variables, without taking into account the acceleration due to gravity g which is a constant, such as:

$$\Phi(y_c; y_n; m; S_0; \varepsilon; \nu) = 0 \tag{12}$$

The graphical representation of the function (12) is not easy, but the particular case of the smooth triangular-shaped channel can facilitate the study of the function. This case will be considered in one of the following sections

GENERAL CRITICAL FLOW RELATIONSHIP

By writing $y_n = y_c$ in Eq. (10), this leads to the general relationship which governs the critical flow in the triangular-shaped channel. The final result is:

$$1 = -\frac{4\sqrt{2}}{\left(1+m^{-2}\right)^{1/4}}\sqrt{S_0}\log\left[\frac{\varepsilon}{7.4y_c}(\sqrt{1+m^{-2}}) + \frac{10.04}{R^*}\right] \tag{13}$$

Introducing Eq.(11) into Eq.(13) results in:

$$1 = -\frac{4\sqrt{2}}{\left(1+m^{-2}\right)^{1/4}}\sqrt{S_0}\log\left[\frac{\varepsilon}{7.4y_c}(\sqrt{1+m^{-2}}) + \frac{0.6275(1+m^{-2})^{3/4}}{\sqrt{g\,y_c^3 S_0}\;/\,\nu}\right] \tag{14}$$

Eq.(14) can be rewritten as:

$$10^{-\frac{(1+m^{-2})^{1/4}}{4\sqrt{2}\sqrt{S_0}}} = \frac{\varepsilon}{7.4y_c}(1+m^{-2})^{1/2} + \frac{0.6275(1+m^{-2})^{3/4}}{\sqrt{g\,y_c^3 S_0}\;/\,\nu} \tag{15}$$

Or:

$$\frac{7.4}{\varepsilon(1+m^{-2})^{1/2}}10^{-\frac{(1+m^{-2})^{1/4}}{4\sqrt{2}\sqrt{S_0}}} = \frac{1}{y_c} + \frac{4.6435(1+m^{-2})^{1/4}}{y_c^{3/2}\varepsilon\left(\sqrt{gS_0}\;/\,\nu\right)} \tag{16}$$

Multiplying the two sides of Eq.(16) by $y_c^{3/2}$, one may obtain:

$$\frac{7.4}{\varepsilon(1+m^{-2})^{1/2}}10^{-\frac{(1+m^{-2})^{1/4}}{4\sqrt{2}\sqrt{S_0}}}\,y_c^{3/2} = y_c^{1/2} + \frac{4.6435(1+m^{-2})^{1/4}}{\varepsilon\left(\sqrt{gS_0}\;/\,\nu\right)} \tag{17}$$

Let's define the following parameters:

$$\varphi_1 = \frac{7.4}{\varepsilon(1+m^{-2})^{1/2}}10^{-\frac{(1+m^{-2})^{1/4}}{4\sqrt{2}\sqrt{S_0}}} \tag{18}$$

$$\varphi_2 = \frac{4.6435(1+m^{-2})^{1/4}}{\varepsilon\left(\sqrt{gS_0}\;/\,\nu\right)} \tag{19}$$

Eq.(17) can be then expressed in the following form:

$$\varphi_1 y_c^{3/2} - y_c^{1/2} - \varphi_2 = 0 \tag{20}$$

The parameters φ_1 and φ_2 are well known, provided *m*, ε, ν and S_0 are given.

By adopting the following change of variables $X = y_c^{1/2}$; Eq. (20) is reduced to:

$$\varphi_1 X^3 - X - \varphi_2 = 0 \tag{21}$$

Eq. (21) is a cubic equation without a second-order term. It can be easily solved for *X* by the circular or hyperbolic trigonometry (Anglin and Lambek, 1995). Once the real value of *X* has been calculated, the required critical depth is then $y_c = X^2$ according to the change of variables previously adopted.

SPECIAL CASE OF A SMOOTH TRIANGULAR-SHAPED CANAL

Variation of the critical depth versus the normal depth

In order to simplify both the study and the calculation, consider the case of a smooth triangular-shaped channel ($\varepsilon \rightarrow 0$), Eq.(10) becomes then:

$$y_c^{5/2} = -4\sqrt{2}\left(1+m^{-2}\right)^{-1/4} y_n^{5/2}\sqrt{S_0}\,\log\left(\frac{10.04}{R^*}\right) \tag{22}$$

The shear Reynolds number R^* is always governed by Eq.(11), i.e.:

$$R^* = 16\frac{\sqrt{g\,y_n^3 S_0}}{\nu}(1+m^{-2})^{-3/4} \tag{11}$$

Introducing Eq.(11) into Eq.(22) and simplifying results in:

$$y_c^{5/2} = -4\sqrt{2}\left(1+m^{-2}\right)^{-1/4} y_n^{5/2}\sqrt{S_0}\,\log\left[\frac{0.6275(1+m^{-2})^{3/4}}{\sqrt{g\,y_n^3 S_0}\,/\,\nu}\right] \tag{23}$$

Eq.(23) is plotted in Fig.(1) showing, for various slopes S_0, the variation of the critical depth y_c with respect to the normal depth y_n for a smooth triangular-shaped channel of side slope $m = 1$ corresponding to an angle $\theta = 45°$ or to an apex angle of 90°. The flowing water is of kinematic viscosity $10^{-6}\,m^2/s$.

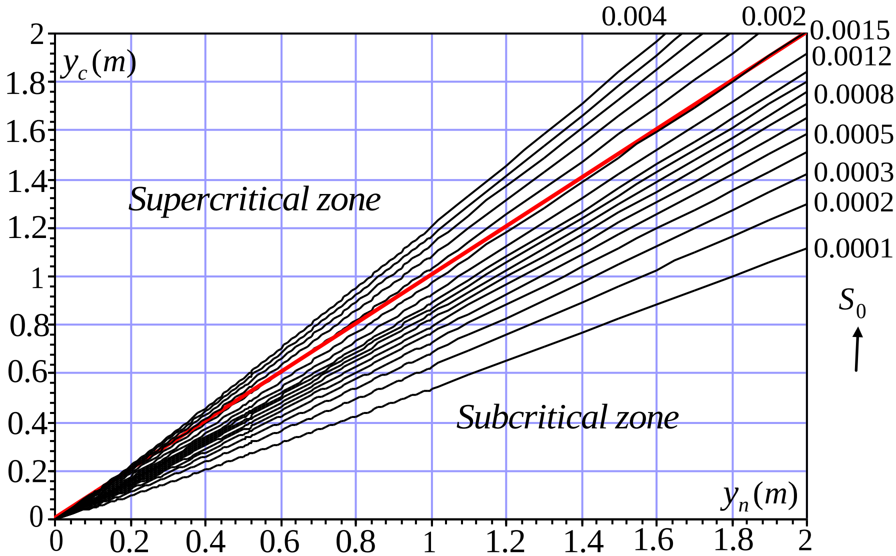

Figure 2: Variation of the critical depth y_c with respect to the normal depth y_n for a smooth triangular-shaped channel of side slope *m* = 1 according to Eq.(23). Red curve: $y_n = y_c$

Figure.2 shows two zones of flow regime namely, a subcritical zone for which y_c is less than y_n, and a supercritical zone which corresponds to y_c greater than y_n. All the curves in Fig. (1) originate in the subcritical zone. Some of them intersect the first bisector at a point which corresponds to $y_n = y_c$, and then end in the supercritical zone. This is the case in particular for the curves corresponding to the slopes S_0 greater than or equal to 0.0012 for which the critical depth is of 5.06 m according to Eq.(23). The curves which correspond to the slopes S_0 below this value do not have a point of intersection with the first bisector. This means that for these slopes, the channel is not the seat of a critical flow (Fig.2). It is worth noting that this finding needs to be completed. In fact, the curves of the subcritical zone intersect the first bisector at a point where the critical depth y_c is extremely high. It is so high that its order of magnitude goes beyond the practical context. As an example, if we consider the case of the slope S_0 = 0.0005, the calculation leads to $y_c = 1130$ meters, which is a practical aberration and a weirdness that does not correspond to any reality. It can therefore be stated without any ambiguity that for the range of values of the flow rates used in practice there are slopes S_0 which do not generate any critical flow in the considered triangular-shaped channel. Fig. (1) also shows that the more the slope S_0 increases, the more the point of intersection with the first bisector approaches shallow depths. As an example, for the slope S_0 = 0.0015 the critical depth is y_c = 1.758 meter, while for the slope S_0 = 0.002 the critical depth is y_c = 0.523 meter.

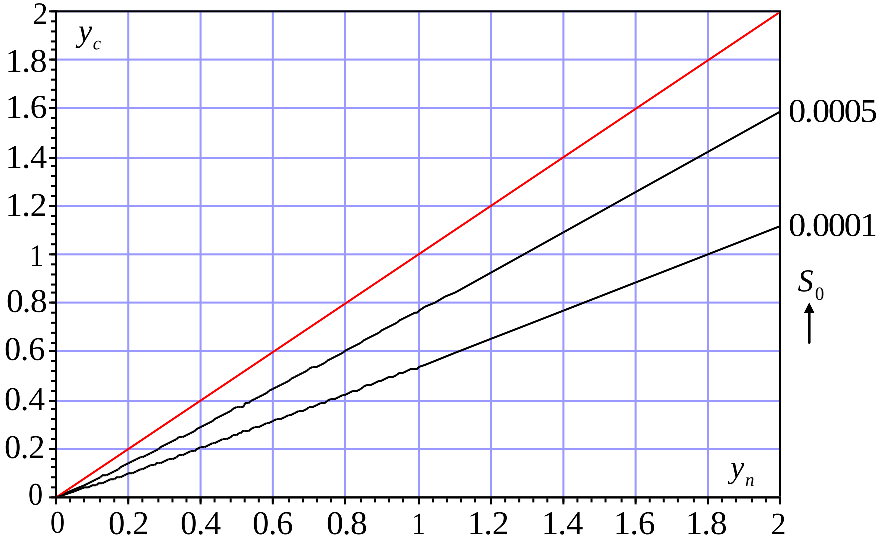

Figure 3: Case of some slopes that do not generate any critical state of the flow in a smooth triangular-shaped channel (*m* = 1). Red curve: $y_n = y_c$

Critical flow depth relationship

The critical depth relationship for the smooth triangular-shaped channel is obtained when writing $y_n = y_c$ in Eq.(23). This then gives:

$$1 = -4\sqrt{2}\left(1+m^{-2}\right)^{-1/4}\sqrt{S_0}\,\log\left[\frac{0.6275\left(1+m^{-2}\right)^{3/4}}{\sqrt{g\,y_c^3 S_0}\,/\,\nu}\right] \tag{24}$$

After some rearrangements, Eq.(24) becomes :

$$y_c = 0.733\left(\frac{\nu^2}{gS_0}\right)^{1/3}\left(1+m^{-2}\right)^{1/2} 10^{\frac{(1+m^{-2})^{1/4}}{6\sqrt{2}\sqrt{S_0}}} \tag{25}$$

Eq.(25) is explicit for the critical depth sought y_c, provided all other parameters are given namely: the slope S_0 of the bottom channel, the side slope m, and the kinematic viscosity ν. The calculations made according to Eq. (25) have shown that, for a given smooth triangular-shaped channel, the critical depth decreases as the slope S_0 increases. In addition, it was observed that, for the same slope S_0, the critical depth decreases as the side slope m increases, i.e. when θ decreases or when the apex angle increases.

Note that in Eq.(25) the quantity $L = [\nu^2/(gS_0)]^{1/3}$ have the dimension of a length. It is therefore relevant to define L as the characteristic length of the critical flow in a smooth triangular-shaped channel.

On the other hand, considering the critical depth y_c as the characteristic length, i.e. $L = y_c$, Eq. (24) is reduced to:

$$R_c^* = 0.6275(1+m^{-2})^{3/4} 10^{\frac{(1+m^{-2})^{1/4}}{4\sqrt{2}\sqrt{S_0}}} \tag{26}$$

The subscript "c" denotes the critical state of the flow. R_c^* is the critical shear Reynolds number expresses as:

$$R_c^* = \frac{\sqrt{g\, y_c^3 S_0}}{\nu} \tag{27}$$

Therefore, for a given smooth triangular-shaped channel, i.e. m and S_0 are known parameters, the critical shear Reynolds number R_c^* is constant.

SPECIAL CASE OF A ROUGH TRIANGULAR-SHAPED CANAL

One may deduce from Eq.(10) the relation between the critical depth y_c and the normal depth y_n for the rough triangular-shaped channel when writing $R^* \to \infty$. This yields:

$$y_c^{5/2} = -4\sqrt{2}\, \frac{y_n^{5/2}}{\left(1+m^{-2}\right)^{1/4}} \sqrt{S_0} \log\left[\frac{\varepsilon}{7.4 y_n} (\sqrt{1+m^{-2}}) \right] \tag{28}$$

On the other hand, Eq.(28) allows deducing the critical depth relationship for a rough triangular-shaped channel when writing $y_n = y_c$. Whence, after some simplifications:

$$1 = -\frac{4\sqrt{2}}{\left(1+m^{-2}\right)^{1/4}} \sqrt{S_0} \log\left[\frac{\varepsilon}{7.4 y_c} (\sqrt{1+m^{-2}}) \right] \tag{29}$$

Eq.(29) can be rewritten as :

$$-\frac{\left(1+m^{-2}\right)^{1/4}}{4\sqrt{2}\sqrt{S_0}}=\log\left[\frac{\varepsilon}{7.4y_c}(\sqrt{1+m^{-2}})\right] \tag{30}$$

Thus:

$$10^{-\frac{\left(1+m^{-2}\right)^{1/4}}{4\sqrt{2}\sqrt{S_0}}}=\frac{\varepsilon}{7.4y_c}(\sqrt{1+m^{-2}}) \tag{31}$$

The critical depth is then governed by the following equation:

$$y_c=\frac{\varepsilon}{7.4}\sqrt{1+m^{-2}}\,10^{\frac{\left(1+m^{-2}\right)^{1/4}}{4\sqrt{2}\sqrt{S_0}}} \tag{32}$$

It is worth noting, on one hand, that for a given rough triangular-shaped channel, the critical depth is directly related to the absolute roughness. On the other hand, for a given rough triangular-shaped channel, i.e. m and S_0 are known parameters, the ratio of the critical depth to the absolute roughness is a constant, i.e. y_c/ε = constant.

The slant length at the critical state of the flow can be expressed as:

$$\ell_c=y_c\sqrt{1+m^2} \tag{33}$$

Combining Eqs.(32) and (33) results in:

$$\ell_c=\frac{\varepsilon}{7.4}\frac{\left(1+m^2\right)}{m}10^{\frac{\left(1+m^{-2}\right)^{1/4}}{4\sqrt{2}\sqrt{S_0}}} \tag{34}$$

Therefore, for a given rough triangular-shaped channel whose the side slope m and the slope S_0 are known, the ratio of the slant length to the absolute roughness is constant, i.e. ℓ_c/ε = constant, or ε/ℓ_c = constant. Given that the wetted perimeter at the critical state is P_c = $2\ell_c$ then one may conclude that the ratio ε/P_c is also constant.

CONCLUSIONS

The functional $\lambda(y_c;y_n;m;S_0;\varepsilon;\nu) = 0$ relationship which links the characteristics of critical and normal flows in a triangular-shaped channel has been theoretically established [Eq. (10)],. where y_c is the critical depth, m is the side slope, S_0 is the channel bottom slope, ε is the absolute roughness, and ν is the kinematic viscosity of the flowing water. This relation was deduced from the general equation valid for all the shapes of channels, established by the authors during a previous study. Given the large number of variables, the graphical representation of the function is not easy. To simplify its study, the special case of the smooth triangular-shaped channel with an opening angle of 90° was considered. For this case, it has been observed that there are S_0 channel slopes which generate critical depths whose values are outside the practical context. It has been estimated that the slope $S_0 = 0.0012$, for instance, generated a critical depth of more than 5m. The more the slope S_0 decreases below this value, the more the critical depth increases in a significant magnitude. The critical depths of a practical order of magnitude are thus obtained for slopes greater than the slope indicated above.

The general relationship which governs the critical flow in a triangular channel has been deduced from the λ function, previously defined, by writing $y_n = y_c$ [Eq.(14)]. Observations and appropriate mathematical manipulations have revealed that y_c is governed by a cubic equation without a second order term [Eq.(21)]. Its analytical resolution is uncomplicated and effortless when one uses circular trigonometry or preferably hyperbolic functions. An application of this relationship to the special cases of both the smooth and the rough triangular-shaped channels ended the present study [Eqs.(25) and (32)].

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