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CONTRIBUTION TO THE STUDY OF THE ISENTROPIC FLOW IN THE LAVAL NOZZLE

CONTRIBUTION A L'TUDE DE L'ECOULEMENT ISENTROPIQUE DANS LA TUYERE DE LAVAL

ACHOUR B.[1], AMARA L.[1,2]

[1] Professor, Research laboratory in subterranean and surface hydraulics (LARHYSS), University of Biskra, Algeria.
[2] Associate Professor, Department of Civil Engineering and Hydraulics, Faculty of Science and Technology, University of Jijel, Algeria.

bachir.achour@larhyss.net

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ABSTRACT

The isentropic flow in the Laval nozzle has been the subject of investigations and theoretical development. In the literature, the flow parameters are expressed as a function of the Mach number M calculated in any cross-sectional area of the flow. In the present study, the flow parameters such as the cross-sectional area A related to the sonic cross-sectional area A_*, the mean velocity V of the flow and mass flow rate are expressed as a function of the flow conditions of the generator state. The important relationship between the Mach number M_* in the sonic section versus the Mach number M in any cross-sectional area of the flow in the nozzle is also presented. Its graphic representation makes it possible to discuss the behaviour of the flow in the nozzle. On the other hand, the normal shock zone has been examined from a theoretical point of view, the development of which has led to the relations which govern the flow parameters in this zone.

The study first concerned the case where the velocity of the flow, subsonic upstream, becomes sonic in the narrowed section of the nozzle and for which the temperature, the pressure and the volume mass are critical. The flow immediately changes to supersonic and to become again further downstream in subsonic by the intermediary of a shock wave. For this case, a detailed numerical example is considered, showing the procedure to be followed to solve the problem.

Through a practical numerical example, the case where the flow remains subsonic over the entire length of its path is considered. It was introduced in this case study the concept of the cross-sectional area of the hypothetical section where the velocity would become sonic and where the temperature, pressure, and density would reach their critical values.

The study ends with the examination of the borderline cases of the flow in the nozzle, as well as the criteria of classification of the flow.

The most important established relationships are represented graphically and interesting conclusions are drawn.

Keywords: Fluid mechanics, fluid flows, parameterization, compressible flow, Laval Nozzle, shock wave.

RESUME

L'écoulement isentropique dans la tuyère de Laval a fait l'objet d'investigations et de développements théoriques. Dans la littérature, les paramètres d'écoulement sont exprimés en fonction du nombre de Mach M calculé en tout point de l'écoulement. Dans la présente étude, les paramètres d'écoulement tels que l'aire de toute section A relative à la section sonique A_*, la vitesse moyenne V de l'écoulement et le débit massique sont exprimés en fonction des conditions d'écoulement à l'état générateur. La relation importante entre le nombre de Mach M_* dans la section sonique et le nombre de Mach M dans toute zone de section transversale de l'écoulement dans la tuyère est également présentée. Sa représentation graphique permet de discuter le comportement de l'écoulement dans la tuyère. D'autre part, la zone de l'onde de choc droite a été examinée d'un point de vue théorique dont le développement a mené aux relations qui gouvernent les paramètres de l'écoulement dans cette zone.

L'étude a d'abord concerné le cas où la vitesse de l'écoulement, subsonique à l'amont, devient sonique dans la section rétrécie de la tuyère et pour laquelle la température, la pression et la masse volumique sont critiques. L'écoulement change aussitôt en supersonique et pour redevenir plus loin à l'aval en subsonique par l'intermédiaire d'une onde de choc. Pour ce cas, un exemple numérique détaillé est considéré, montrant la démarche à suivre pour résoudre le problème.

A travers un exemple numérique pratique, le cas où l'écoulement reste subsonique sur toute la longueur de son trajet. Il a été introduit dans cette étude de cas le concept de l'aire de la section hypothétique où la vitesse deviendrait sonique et où la température, la pression et la masse volumique atteindraient leurs valeurs critiques.

Les relations établies les plus importantes sont représentées graphiquement et des conclusions intéressantes sont tirées.

Mots clés : Mécanique des fluides, écoulements des fluides, paramétrisation, écoulement compressible, Tuyère de Laval, onde de choc.

INTRODUCTION

A nozzle is a passive mechanical component as an hourglass-shaped tube that places two reservoirs at different pressures in communication and whose profile must theoretically allow a reversible adiabatic flow. The nozzle profile must therefore be such that it exactly dresses the flow vein. Thus, for example, if the pressure varies in the nozzle from the value P_1 (Figure 1) to a zero value in the direction of flow, the profile must be convergent, and then diverge and the outlet section must be infinite. The entry section will itself be infinite if the entry velocity is zero (Figure 1).

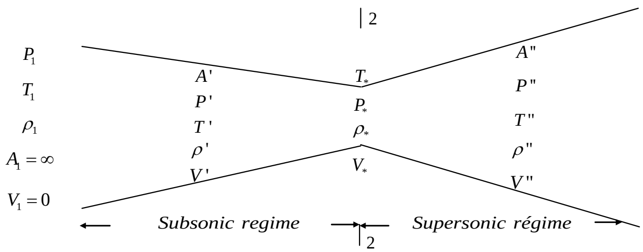


Figure 1: Characteristics of the flow in the Laval nozzle for some cross-sectional areas

The Laval nozzle is used to accelerate hot, pressurized gases that pass through it until they reach supersonic velocity. The nozzle optimally converts the heat of the gases into kinetic energy. It makes it possible to produce large amounts of energy from combustion gases. Laval nozzles are used in rocket engines, steam turbines and gas turbines. In the case of a rocket engine, this type of nozzle plays a fundamental role in optimizing thrust by maximizing the velocity of gas ejection.

In practice, the theoretical flow conditions are never observed and, in particular, the viscosity of the fluid is not zero, which leads to irreversibilities. Depending on the expansion rate used, the nozzles are either simply convergent for high expansion rates $\delta > \delta_0$, or convergent-divergent (Figure 1) when the expansion rates are less than δ_0 (Lallemand, 2020). They are then called Laval nozzles in homage to Carl Gustaf Patrik de Laval (1845-1913). In a Laval nozzle, the pressure varies from a high pressure P_1 to an outlet pressure which can be relatively low.

In the Laval nozzle, the diameter begins by reducing in the direction of gas circulation, then increases again. It comprises three parts namely: the convergent which is the part of the nozzle which narrows, the throat is the cross-sectional of the nozzle where the diameter is minimum, and the divergent whose diameter increases again (Li, 2008).

For a Laval nozzle to accelerate gases optimally, it is necessary that the convergent and the divergent (which are not symmetrical) have very precise shapes and that the diameter of the throat takes a given value. All these parameters are determined from the characteristics of the incoming gas (pressure, temperature, flow rate, molecular mass) and from the external pressure.

The principle of operation of a Laval nozzle is based on the properties of gases when they circulate at subsonic and supersonic velocities. When gas flows at a subsonic velocity through a pipe with a narrowing diameter, its velocity increases. However, the velocity of gas cannot exceed that of sound, i.e. Mach number equal to1. In fact, in a supersonic flow regime, i.e. the velocity greater than the velocity of sound, the behaviour of the gas is reversed: for its velocity to increase, the diameter of the pipe must increase as shown by the Hugoniot relationship (Comolet, 1969; Landau and Lifchits, 1987). To accelerate a gas to supersonic velocity, it is therefore necessary that it first circulate in a converging section of pipe until it reaches the velocity Mach 1 in the throat section and from this section the gas must progress in a pipe of increasing diameter, the expanded part, so that the velocity continues to increase. The Laval nozzle only works according to this principle if the gas velocity reaches Mach 1 velocity at the throat. To achieve this, the nozzle must be designed so that the outlet pressure is at least two times lower than that at the inlet. If this condition is met, the velocity at the throat reaches Mach 1 and the nozzle is said to be primed. If the outlet pressure is greater than this value, the nozzle will not prime. On the contrary, if the pressure ratio is greater, the yield increases. This is optimal when the outlet pressure is equal to the ambient pressure (at ground level, i.e.1 bar): we then say that the nozzle is suitable. For a rocket engine, the cross--sectional area ratio of the expanded part must therefore be all the more important when the engine operates at high altitudes, that is to say at low ambient pressures (Anderson Jr., 1991).

In the Laval nozzle, the flow is isentropic, i.e. the entropy is constant, a consequence of the assumption that it is a non-viscous fluid and that the process is adiabatic, i.e. there is no heat exchange between the fluid and the nozzle. The gas which circulates in the nozzle is compressible. Several relationships can be established to define the pressure, density and temperature of the isentropic flow of an ideal gas. They are often called isentropic relations. The characteristics of the flow in the nozzle, generally related to those of the flow at the inlet of the nozzle, are expressed as a function of the Mach number M. Although their values are tabulated, the reference literature that the authors consulted does not indicate clearly relationships expressing the ratio of the cross-sectional areas A/A_* as a function of the ratios of temperatures T/T_1, of the ratio of densities ρ/ρ_1 and of the ratio of pressures P/P_1, where A is any cross-sectional area, A_* is the sonic cross-sectional area and ρ is the density of the gas (Figure 1). The characteristics having the index "1" are those referred to the inlet of the nozzle. All of these relationships, among others, will be established in this study, including the important relationship that links the Mach number M_* in the throat to the Mach number M at any point.

The study will focus in particular on the flow in the presence of a normal shock wave in supersonic conditions. The study will not focus on the shock zone itself (Azdasher, 2012; Comolet, 1985; Candel, 1995).

All the important established relationships will be represented graphically and interesting conclusions will be drawn.

FUNDAMENTAL RELATIONSHIPS

The ideal gas law can be written as (Oswatitsch, 1956):

$$\frac{P}{\rho} = gRT \tag{1}$$

Where P (Pa or kg.m^{-1}.s^{-2}) is the pressure, ρ is the density (kg.m^{-3}), g is the acceleration due to gravity (m.s^{-2}), R is the universal ideal gas constant which is expressed in relation (1) in the units m/°K (meters by degrees Kelvin) and T is the absolute temperature (°K). Some authors prefer to use the following ideal gas relationship: $P/\rho = RT$ and in this case the constant R is expressed in J.K^{-1}.mol^{-1}. The universal ideal gas constant R is related to the molar mass M of the gas by the following relation:

$$R = \frac{848}{\mathrm{M}} \tag{2}$$

The isentropic laws governing the characteristics of the gas are such as (Rathakrishnan, 2019):

$$\frac{P_1}{\rho_1^k} = \frac{P}{\rho^k} = \text{constant} \tag{3}$$

$$\frac{P}{P_1} = \left(\frac{\rho}{\rho_1}\right)^k = \left(\frac{T}{T_1}\right)^{\frac{k}{k-1}} \tag{4a}$$

$$\frac{\rho}{\rho_1} = \left(\frac{P}{P_1}\right)^{1/k} = \left(\frac{T}{T_1}\right)^{1/(k-1)} \tag{4b}$$

$$\frac{T}{T_1} = \left(\frac{P}{P_1}\right)^{(k-1)/k} = \left(\frac{\rho}{\rho_1}\right)^{k-1} \tag{4c}$$

In these relations the constant k is a characteristic of the gas. It is called the adiabatic constant or Laplace coefficient and sometimes the compressibility coefficient. It is often indicated in the literature by the symbol γ. It has no unity because it is expressed by the ratio between the specific heat at constant pressure c_p and the specific heat at constant volume c_V. For monoatomic gases: $k = (5/2)/(3/2) \approx 1.67$, for diatomic gases: $k = (7/2)/(5/2) = 1.4$ and for polyatomic gases: $k \approx 1.3$.

The continuity equation is expressed as:

$$V\rho A = V_1\rho_1 A_1 = V_n\rho_n A_n = \text{constant} \tag{5}$$

Where V (m.s^{-1}) is the average velocity of the flow and A (m^2) is the cross-sectional area. The subscript "n" denotes any vertical section n-n. Eq. (5) expresses the mass flow rate denoted $\dot{m}$.

The energy equation per unit mass of frictionless isentropic compressible flows is given by the following relation (Zucrow and Hoffman, 1976):

$$gz_1 + \alpha_1 \frac{V_1^2}{2} + \frac{k}{k-1}\frac{P_1}{\rho_1} = gz + \alpha \frac{V^2}{2} + \frac{k}{k-1}\frac{P}{\rho} \quad (6a)$$

z is the position of the center of gravity of the considered section, with respect to a horizontal reference plane.

The energy per unit weight equation of frictionless isentropic compressible flows is written as:

$$z_1 + \alpha_1 \frac{V_1^2}{2g} + \frac{k}{k-1}\frac{P_1}{\rho_1 g} = z + \alpha \frac{V^2}{2g} + \frac{k}{k-1}\frac{P}{\rho g} \quad (6b)$$

The equation of energy per unit mass of isothermal compressible flows is:

$$gz_1 + \alpha_1 \frac{V_1^2}{2} + gRT \log P_1 = gz + \alpha \frac{V^2}{2} + gRT \ln P + \Delta E \quad (6c)$$

ΔE is the loss of mechanical energy per unit mass converted into heat. Generally: $z = z_1$ and $\alpha = \alpha_1 = 1$

Eq.(6a) then becomes:

$$\frac{V_1^2}{2} + \frac{k}{k-1}\frac{P_1}{\rho_1} = \frac{V^2}{2} + \frac{k}{k-1}\frac{P}{\rho} \quad (6d)$$

The actual or hypothetical sonic velocity passing through the nozzle is:

$$c_* = \sqrt{\frac{2k}{k+1} gRT_1} = \sqrt{\frac{2k}{k+1}\frac{P_1}{\rho_1}} \quad (7)$$

Relation (7) was obtained by the authors by considering the temperature ratio $T_1 / T = f(k; M)$ given by the literature where the Mach number M = 1 and $T = T_*$ representing the temperature in the narrowed section of the nozzle. Relation (8a) below follows from these considerations.

Critical ratios of gas characteristics are (Rathakrishnan, 2019):

$$\frac{T_*}{T_1} = \frac{2}{k+1} \tag{8a}$$

$$\frac{P_*}{P_1} = \left(\frac{2}{k+1}\right)^{k/(k-1)} \tag{8b}$$

$$\frac{\rho_*}{\rho_1} = \left(\frac{2}{k+1}\right)^{1/(k-1)} \tag{8c}$$

By taking into account relation (7) and relations (8), it can be seen that the value c_* of the sonic velocity as well as the critical values P_*, T_* and ρ_* corresponding respectively to the pressure, the absolute temperature and the density are well determined by the only conditions upstream of the flow in the nozzle and whose characteristics have the index "1".

The equation of momentum applied to the shock zone, assuming on the one hand that the momentum correction factor is $\beta = 1$ and on the other hand the thickness of the shock zone is very small, i.e. $A_3 = A_4$, is written as:

$$P_3 - P_4 = \rho_4 V_4^2 - \rho_3 V_3^2 \tag{9}$$

CONDITIONS IN THE REDUCED SECTION

The flow regime passing through the narrowed section 2-2 of Figure 1 is:

1. Sonic, if the ratios T_2 / T_1, P_2 / P_1 and ρ_2 / ρ_1 are identified with the critical ratios defined by equations (8a); (8b) and (8c).

2. Subsonic, if the following ratios T_2 / T_1, P_2 / P_1, and ρ_2 / ρ_1 have values greater than the critical ratios. In this case, the flow retains its subsonic character all along the nozzle, in the convergent, in the constricted section and also downstream thereof.

FLOW VELOCITY RELATIONSHIPS

To determine the relations governing the flow velocity V in any arbitrarily chosen section of an isentropic, subsonic or supersonic flow, equation (6a) must be applied between the chosen section and section 1-1. The velocity in section 1-1 being such that $V_1 = 0$ and knowing that $z = z_1$, equation (6a) becomes:

$$\frac{k}{k-1}\frac{P_1}{\rho_1} = \frac{V^2}{2} + \frac{k}{k-1}\frac{P}{\rho} \tag{6d}$$

Taking into account Eq. (1), the previous equation (6d) becomes:

$$\frac{k}{k-1} gRT_1 = \frac{V^2}{2} + \frac{k}{k-1} gRT$$

Hence:

$$V = \sqrt{\frac{2k}{k-1} gRT_1 (1 - T/T_1)} \tag{10a}$$

Or, taking into account equation (4a), one gets:

$$V = \sqrt{\frac{2k}{k-1} gRT_1 \left[1 - \left(P/P_1\right)^{(k-1)/k}\right]} \tag{10b}$$

Considering Eq.(4c), one may then write:

$$V = \sqrt{\frac{2k}{k-1} gRT_1 \left[1 - \left(\rho/\rho_1\right)^{(k-1)}\right]} \tag{10c}$$

MASS FLOW RATE RELATIONSHIPS

Taking into account the continuity equation expressed by relation (5) as well as equations (4a), (4b) and (4c), we can deduce that the mass flow rate is expressed by the following equations:

$$\dot{m} = A\rho_1 \left(\frac{T}{T_1}\right)^{1/(k-1)} \sqrt{\frac{2k}{k-1} gRT_1 \left(1 - T/T_1\right)} \tag{11a}$$

$$\dot{m} = A\rho_1 \left(\frac{P}{P_1}\right)^{1/k} \sqrt{\frac{2k}{k-1} gRT_1 \left[1 - \left(\frac{P}{P_1}\right)^{(k-1)/k}\right]} \tag{11b}$$

$$\dot{m} = A\rho\sqrt{\frac{2k}{k-1} gRT_1 \left[1-\left(\rho / \rho_1\right)^{(k-1)}\right]} \tag{11c}$$

We can form the dimensionless number $\dot{\mathrm{M}}$ such that:

$$\dot{\mathrm{M}} = \frac{\dot{m}}{A\rho_1\sqrt{gRT_1}} = \left(\frac{T}{T_1}\right)^{1/(k-1)} \sqrt{\frac{2k}{k-1}\left(1-T/T_1\right)} = \phi(T/T_1) \tag{11d}$$

It can therefore be seen that the relative mass flow rate increases with the decrease of the cross-sectional area *A*, which is the case with the converging part of the nozzle. Conversely, the relative flow rate decreases when *A* increases, which is the case with the expanded part of the nozzle. It is therefore logical to assume that the relative mass flow passes through a maximum.

The in-depth study of the relation (11d) as well as its graphical representation (Figure 2) showed that, for k = 1.4, $\dot{\mathrm{M}}$ increases up to a maximum as T/T_1 decreases. This can be seen on the right branch in Figure.2. It variation begins from zero value for $T/T_1 = 1$. The calculation has shown that $\dot{\mathrm{M}}_{\max} = 0.68473146 \approx 0.685$ for $T/T_1 = 5/6 = 0.8333$. This value corresponds to the value of the critical temperature ratio $T_*/T_1 = 2/(k+1)$ expressed by the relation Eq. (8a) for k =1.4. This therefore means that, for diatomic gases, if $T_* = 5T_1/6$, then the relative mass flow rate $\dot{\mathrm{M}}$ is maximum. Just as we can show, through the study of relations (11b) and (11c), that M is maximal when $P_* \approx 0.5283P_1$ and $\rho_* \approx 0.634\rho_1$. The relative mass flow rate decreases from the maximum until reaching zero for $T/T_1 = 0$, as shown on the left branch of the curve in Figure3. Finally, one may write what follows:

$$\dot{m}_{\max} \approx 0.685 A_* \rho_1 \sqrt{gRT_1} \tag{11e}$$

To find the result $T/T_1 = 2/(k+1)$, write $d\dot{M}/(T/T_1) = 0$ in Eq. (11d). After derivation, it comes that:

$$\sqrt{\frac{2k}{k-1}}\left[\frac{1}{k-1}\left(T/T_1\right)^{(2-k)/(k-1)}\left(1-T/T_1\right)^{1/2} - \frac{1}{2}(T/T_1)^{1/(k-1)}\left(1-T/T_1\right)^{-1/2}\right] = 0$$

After some simplifications and rearrangements, one may get:

$$\frac{2}{k-1}(T/T_1)^{-1}(1-T/T_1)-1=0$$

It is then easy to show that:

$$T/T_1 = 2/(k+1)$$

Inserting the previous result into Eq. (11d) result in:

$$\dot{\mathrm{M}}_{\max} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \sqrt{k} \qquad (11f)$$

For k =1.4, Eq. (11f) gives $\dot{\mathrm{M}}_{\max} = 0.68473146 \approx 0.685$.

According to the definition of the relative mass flow rate, one may write:

$$\dot{m}_{\max} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \sqrt{k}\, A_* \rho_1 \sqrt{gRT_1} \qquad (11g)$$

This is the general relationship of the maximum mass flow rate. The maximum mass flow rate is reached when the gas velocity reaches sonic velocity in the throat section of the nozzle. Under these conditions the flow is choked (Thermopedia, 2020). The throat size is chosen to choke the flow and set the mass flow rate through the system. The flow in the throat is sonic which means the Mach number is equal to one in the throat.

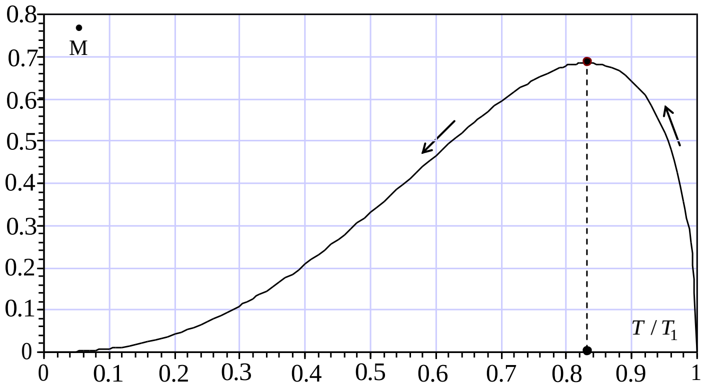


Figure 2: Variation of the relative mass flow rate as a function of the temperature ratio according to Eq. (11d).

NOTE

The right branch of the curve in figure (2) reflects the variation in the relative mass flow rate in the converging part of the nozzle, while the left branch concerns the expanded part of the nozzle. The symbol (•) on the x-axis correspond to the sonic section where the temperature ratio T/T_1 reaches the critical temperature ratio expressed by Eq. (8a).

According to Eq. (11d), the relative mass flow rate is as:

$$\dot{\mathrm{M}} = \frac{\dot{m}}{A\rho_1\sqrt{gRT_1}} \tag{11d}$$

The previous equation can be rewritten is the following form:

$$\dot{\mathrm{M}} = \frac{\dot{m}/A_*\rho_1\sqrt{gRT_1}}{A/A_*} \tag{11h}$$

Finally, one may write:

$$\dot{\mathrm{M}} = 0.685\frac{\dot{m}/\dot{m}_{\max}}{A/A_*} \tag{11i}$$

As the mass flow rate is constant, i.e. $\dot{m}/\dot{m}_{\max} = 1$, Eq. (11i) reduces to:

$$\dot{\mathrm{M}} = \frac{0.685}{A/A_*} \tag{11j}$$

Therefore, the relative mass flow rate reaches the value 0.685, as previously determined, when the cross-sectional area ratio A/A_* is equal to unity. The cross-sectional area A corresponds to the cross-sectional area of the throat.

CROSS-SECTIONAL AREA RELATIONSHIPS

Likewise, by applying the continuity equation expressed by relation (5) to the narrowed section and taking into account the relation (11a), it comes that:

$$\dot{m} = c_*\rho_*A_* = A\rho_1\left(\frac{T}{T_1}\right)^{1/(k-1)}\sqrt{\frac{2k}{k-1}gRT_1\left(1-T/T_1\right)}$$

From one we can deduce the cross-sectional area of the narrowed section, after taking into account relations (7) and (8c), such as:

$$A_* = \frac{\overset{\bullet}{m}}{c_*\rho_*} = \frac{A\rho_1\left(\dfrac{T}{T_1}\right)^{1/(k-1)}\sqrt{\dfrac{2k}{k-1}gRT_1\left(1-T/T_1\right)}}{\sqrt{\dfrac{2k}{k+1}gRT_1}\ \rho_1\left(\dfrac{2}{k+1}\right)^{1/(k-1)}}$$

Whence:

$$\frac{A}{A_*} = \frac{\sqrt{\dfrac{2k}{k+1}gRT_1}\ \rho_1\left(\dfrac{2}{k+1}\right)^{1/(k-1)}}{\rho_1\left(\dfrac{T}{T_1}\right)^{1/(k-1)}\sqrt{\dfrac{2k}{k-1}gRT_1\left(1-T/T_1\right)}}$$

By squaring the two members of the equation and after some simplifications, it follows that:

$$\left(\frac{A}{A_*}\right)^2 = \frac{\left(\dfrac{k-1}{k+1}\right)\left(\dfrac{2}{k+1}\right)^{2/(k-1)}}{\left(\dfrac{T}{T_1}\right)^{2/(k-1)}-\left(\dfrac{T}{T_1}\right)^{(k+1)/(k-1)}} = \varphi_T\left(T/T_1\right) \qquad \text{(12a)}$$

It is thus observed that the ratio A/A_* depends only on the temperature ratio T/T_1. If one replaces in relation (12a) the temperature ratio T/T_1 by the critical ratio T_*/T_1 given by relation (8a), it will find that $A/A_*=1$ which corresponds to the area of the sonic section. Conversely, according to relation (12a), the condition $A/A_*=1$ can only be satisfied if $T/T_1= T_*/T_1=2/(k+1)$. Equation (8a) is then reproduced. Likewise, if we replace in relation (10a) the ratio T/T_1 by the critical ratio $T/T_1=T_*/T_1=2/(k+1)$, one ends up with the equality $V = c_*$ and the relation (7) is then reproduced.

Let us take the case of diatomic gases whose adiabatic constant is:

$$k = c_p / c_v = (7/2)/(5/2) = 1.4$$

The relation (12a) becomes:

$$\left(\frac{A}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{T}{T_1}\right)^5 - \left(\frac{T}{T_1}\right)^6}$$

Its graphic representation is shown in Figure 3.

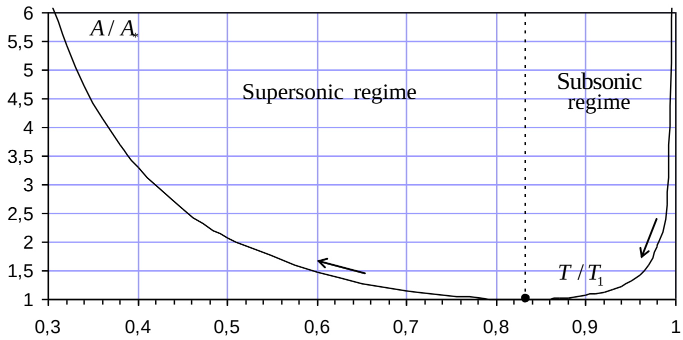


Figure 3: Variation of $A/A_* = f(T/T_1)$ **according to equation (12a) in the case of a diatomic gas with adiabatic constant *k* = 1.4. (•) Sonic section corresponding to** $T_*/T_1 = 5/6 \approx 0.8333$**.**

There are two cross-sectional areas: in the first one the flow is subsonic (converging part) and in the second the flow is supersonic (expanded section). For these two cross-sectional areas, correspond two different ratio values of T/T_1. One can also see that the curve passes through a minimum which corresponds to $T/T_1 = T_*/T_1 = 5/6 \approx 0.8333$ and to $A/A_* = 1$. These are the characteristics of the sonic section. The value $T/T_1 = T_*/T_1 = 5/6$ can be obtained by applying relation (8a) for $k = 1.4$.

Taking into account equation (4c), relation (12a) becomes:

$$\left(\frac{A}{A_*}\right)^2 = \frac{\left(\frac{k-1}{k+1}\right)\left(\frac{2}{k+1}\right)^{2/(k-1)}}{\left(\frac{P}{P_1}\right)^{2/k} - \left(\frac{P}{P_1}\right)^{(k+1)/k}} = \varphi_p(P/P_1) \qquad (12b)$$

Relation (12b) clearly shows that the section ratio A/A_* only depends on the pressure ratio. One can show that in relation (12b), the condition $A/A_* = 1$ corresponding to the sonic section is satisfied for the critical ratio $P/P_1 = P_*/P_1 = \left[2/(k+1)\right]^{k/(k-1)}$.

In the case of diatomic gases for which k = 1.4, the critical pressure ratio is therefore $P_* / P_1 = 0.52828179$.

The relation (12b) has been represented graphically in Figure 4 for the case of diatomic gases with adiabatic constant k = 1.4.

For this value, relation (12b) is written as:

$$\left(\frac{A}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{P}{P_1}\right)^{10/7} - \left(\frac{P}{P_1}\right)^{12/7}}$$

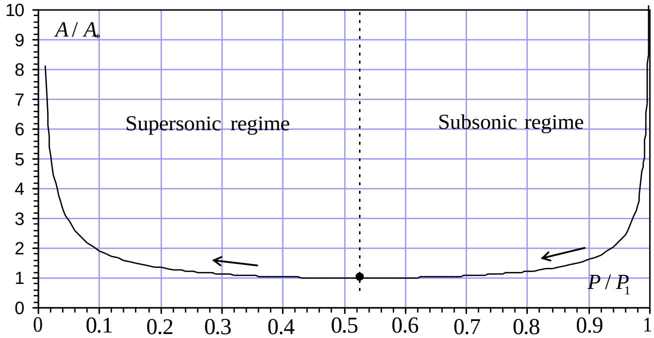


Figure 4: Variation of $A / A_* = f(P / P_1)$ according to equation (12b) in the case of a diatomic gas with adiabatic constant *k* = 1.4. (•) Sonic section $P_* / P_1 \approx 0.5283$

So, there are two sections: in the first one the flow is subsonic (converging part) and in the second one the flow is supersonic (expanded part). These two sections correspond to different ratio values of P / P_1. One can also see that the curve passes through a minimum which corresponds to $P / P_1 = P_* / P_1 \approx 0.52828179$ and to $A / A_* = 1$, as it has had been already mentioned previously. These are the characteristics of the sonic section or narrowed section of the nozzle.

Taking into account Eq. (4a), Eq. (12b) becomes:

$$\left(\frac{A}{A_*}\right)^2 = \frac{\left(\frac{k-1}{k+1}\right)\left(\frac{2}{k+1}\right)^{2/(k-1)}}{\left(\frac{\rho}{\rho_1}\right)^2 - \left(\frac{\rho}{\rho_1}\right)^{(k+1)}} = \varphi_\rho(\rho / \rho_1) \quad (12c)$$

Relation (12c) thus expresses the ratio A/A_* of the sections as a function of the density ratios ρ/ρ_1 exclusively. One can show that, in relation (12c), the condition $A/A_* = 1$ corresponding to the sonic section is satisfied for the critical ratio $\rho/\rho_1 = \rho_*/\rho_1 = \left[2/(k+1)\right]^{1/(k-1)}$ and the relation (8c) is then reproduced. In the case of diatomic gases for which k = 1.4, the critical ratio of the densities is therefore $\rho_*/\rho_1 = 0.63393815$.

For this type of gas, the relation (12c) is written as:

$$\left(\frac{A}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{\rho}{\rho_1}\right)^2 - \left(\frac{\rho}{\rho_1}\right)^{2.4}}$$

This relationship is illustrated graphically in Figure 5

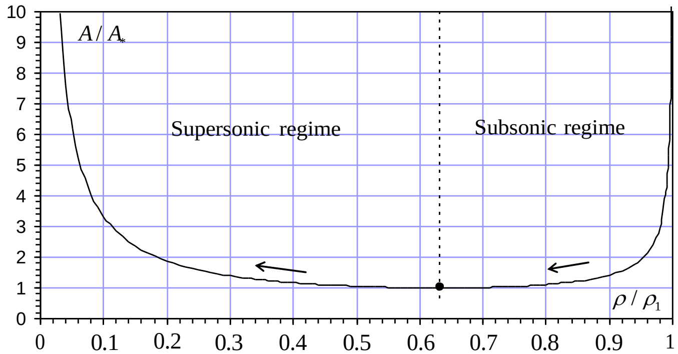


Figure 5: Variation of $A/A_* = f(\rho/\rho_1)$ according to equation (12c) in the case of a diatomic gas with adiabatic constant k = 1.4. (•) Sonic section $\rho_*/\rho_1 \approx 0.634$

There are thus two sections: in the first one the flow is subsonic (converging) and the other where the flow is supersonic (expanded section). For these two sections correspond two different ratio values of ρ/ρ_1.

It can also be seen that the curve passes through a minimum which corresponds to $\rho/\rho_1 = \rho_*/\rho_1 \approx 0.634$ and to $A/A_* = 1$ corresponding to the sonic section.

MIXED FLOW WITH NORMAL SHOCK WAVE

Mach number M_* in the throat

The flow of a gas passing through the nozzle represented by figure 6 undergoes acceleration in the convergent but remains subsonic upstream of the narrowed section 2-2, then it can become sonic in this section or in a fictitious or hypothetical section downstream. The flow then changes to supersonic immediately downstream of section 2-2 and is accelerated between sections 2-2 and 3-3. A shock wave occurs between sections 3-3 and 4-4 beyond which it remains subsonic and decelerates. Parametric relationships in the shock zone will be established and exposed by the authors in the following section

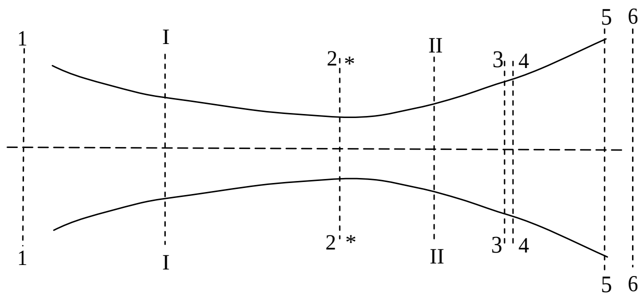


Figure 6: Definition sketch of different cross-sectional areas in the Laval nozzle

The physical laws expressed by relations (1) and (2) as well as the continuity equation (5) are valid in all sections of the flowing gas passing through the nozzle. Isentropic relations (3) and (4), applicable in any section between 1-1 and 3-3 and between sections 4-4 and 6-6, are not valid between sections 3-3 and 4-4 that are the geometrical locus of the shock zone.

To find a solution to the problem exposed above, having a general validity, one must establish the value of the following parametric relations $(V_n / c_*)^2$, $(A_n / A_*)^2$, T_n / T_1, P_n / P_1 and ρ_n / ρ_1, in each of the sections 1-1,2-2,… 6-6. It is also necessary to specify the relations giving the value of the following parametric relations: $(V / c_*)^2$, $(A / A_*)^2$, T / T_1, P / P_1 and ρ / ρ_1 in any section of each of the following zones 1-2, 1-3, 3-4, and 4-5.

In section 1-1, one can write, for obvious reasons: $(V_1 / c_*)^2 = 0$, $(A_1 / A_*)^2 = \infty$, $T_1 / T_1 = P_1 / P_1 = \rho_1 / \rho_1 = 1$. In any section intersecting the upper isentropic zone 1-3, by applying the energy equation (6b) between section 1-1 and any other arbitrarily chosen section, one may write when taking into account the relation (1):

$$V^2 + \frac{2k}{k-1} gRT = \frac{2k}{k-1} gRT_1$$

Hence:

$$V^2 = \frac{2k}{k-1} gRT_1 (1 - T / T_1)$$

By multiplying the right-hand side of the previous equation by the quantity (k + 1) / (k + 1) and dividing by the equation (7) squared, it comes out that:

$$\left(\frac{V}{c_*}\right)^2 = \frac{k+1}{k-1}(1 - T / T_1) \tag{13a}$$

For $V = V_*$, on may recognize, in the left-hand side of the previous equation, the Mach number $M_* = V_* / c_*$ in the throat, that is:

$$M_*^2 = \frac{k+1}{k-1}(1 - T / T_1) \tag{13b}$$

For diatomic gases with a compressibility coefficient k = 1.4, relation (13b) indicates that for $T / T_1 = 0$, $M_* = \sqrt{6} \approx 2.45$, while for $T / T_1 = 1$, one gets $M_* = 0$. One can also note that for $T / T_1 = 5/6 = 0.83333$, the Mach number M_* is such that $M_* = 1$, indicating that the flow becomes sonic in the narrowed section of the nozzle or in a fictitious or hypothetical sonic section downstream of the narrowed section as it is shown in Figure 7 of the discussion section.

Taking into account Eq.(4c), Eq.(13a) can also be written in the following forms:

$$M_*^2 = \frac{k+1}{k-1}\left[1 - (P / P_1)^{(k-1)/k}\right] \tag{13c}$$

$$M_*^2 = \frac{k+1}{k-1}\left[1 - (\rho / \rho_1)^{(k-1)}\right] \tag{13d}$$

The literature gives the following ratios T_1 / T, P_1 / P and ρ_1 / ρ as a function of the Mach number M (Rathakrishnan, 2019). The temperature ratio $T_1 / T = f(M)$ is such that:

$$T_1 / T = 1 + \frac{k-1}{2} M^2 \tag{14}$$

By combining relations (13a) and (14), it is easy to show that:

$$M_* = \frac{(k+1)M^2}{2+(k-1)M^2} \tag{15}$$

One thus obtains the functional relationship $M^* = f(k;M)$. For diatomic gases with a compressibility coefficient k = 1.4, relation (15) becomes

$$M_* = \frac{2.4M^2}{2+0.4M^2} \tag{16}$$

Note some particular values such as $M_* = 0$ for $M = 0$ and $M_* = 1$ for $M = 1$.

Relation (16) has been represented graphically in figure 7.

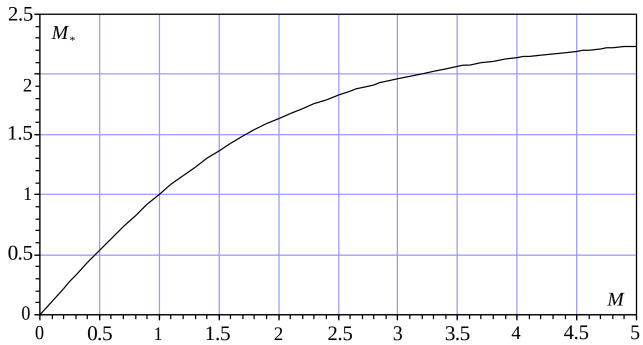


Figure 7: Variation of the Mach number M_* as a function of the Mach number *M* according to equation (16) valid for diatomic gases (*k* = 1.4)

Figure 7 shows that when *M* increases, M_* also increases but this increase is only relative beyond the value of the Mach number *M* = 1. In the range 0 < *M* <1, the Mach number M_* also varies in the range 0 < M_* <1 but with values slightly higher than those of *M*. When the flow is sonic corresponding to *M* = 1, the flow in the narrowed section remains sonic with M_* =1. During the supersonic flow corresponding to *M* > 1, the flow in the narrowed section also remains supersonic with M_* > 1. It can be concluded that the flow regime does not change when passing from the convergent to the narrowed section. However, for values of *M* < 1 corresponding to a subsonic flow in the convergent, there is a fictitious or hypothetical sonic section that can be obtained by extending the convergent over a certain distance as shown in figure 7 of the discussion paragraph. All of the flow parameters in this fictitious or hypothetical sonic section are marked with the index "*".

Parametric relations in the shock zone

As mentioned above, isentropic relations (3) and (4), applicable in any section between 1-1 and 3-3 and between sections 4-4 and 6-6, are not valid between sections 3-3 and 4-4 that are the geometrical locus of the shock zone (Figure 6).

By applying the continuity equation (5) to sections 3-3 and 4-4 and assuming that the distance between these two sections is small, it comes that:

$$A_3 = A_4$$

$$\rho_4 V_4 = \rho_3 V_3 \tag{5}$$

For the same reasons, the momentum equation applied to the said sections is reduced to:

$$P_3 - P_4 = \rho_4 V_4^2 - \rho_3 V_3^2 \tag{9}$$

Combining Eqs. (5) and (9) results in:

$$\frac{P_3}{\rho_3 V_3} - \frac{P_4}{\rho_4 V_4} = V_4 - V_3 \tag{23}$$

On the other hand, by applying the energy equation (6) between a cross-sectional area arbitrarily chosen and the sonic cross-sectional area, one can write:

$$V^2 + \frac{2k}{k-1}\frac{P}{\rho} = c_*^2 + \frac{2k}{k-1}\frac{P_*}{\rho_*}$$

Taking into account Eq. (7), the previous equation becomes:

$$V^2 + \frac{2k}{k-1}\frac{P}{\rho} = c_*^2 + \frac{2}{k-1}c_*^2$$

That is to say:

$$V^2 + \frac{2k}{k-1}\frac{P}{\rho} = \frac{k+1}{k-1}c_*^2$$

Whence:

$$\frac{P}{\rho} = \frac{(k+1)\,c_*^2 - (k-1)V^2}{2k} \tag{24}$$

Applying Eq. (24) to the cross-sectional areas 3-3 and 4-4 results in:

$$\frac{P_3}{\rho_3}=\frac{(k+1)\,c_*^2-(k-1)V_3^2}{2k} \qquad (24a)$$

$$\frac{P_4}{\rho_4}=\frac{(k+1)\,c_*^2-(k-1)V_4^2}{2k} \qquad (24b)$$

Eliminating P_3/ρ_3 and P_4/ρ_4 between equations (23), (24a) and (24b), it follows that:

$$\frac{(k+1)\,c_*^2-(k-1)V_3^2}{2kV_3}-\frac{(k+1)\,c_*^2-(k-1)V_4^2}{2kV_4}=V_4-V_3$$

After some arrangements, the previous relation reduces to:

$$\left(V_4-V_3\right)\left(\frac{k+1}{2k}\,\frac{c_*^2}{V_3V_4}+\frac{k-1}{2k}\right)=V_4-V_3$$

This last equation is satisfied if $V_4=V_3$, that is, if there is no shock wave. The equation is also satisfied if:

$$\frac{k+1}{2k}\,\frac{c_*^2}{V_3V_4}+\frac{k-1}{2k}=1$$

That is to say if:

$$\frac{c_*^2}{V_3V_4}=1$$

Or:

$$V_3V_4=c_*^2 \qquad (25)$$

One can thus recognize Prandtl's relation (Ma and Wang, 2005).

From relation (25) one can deduce that:

$$\frac{V_4}{c_*}=\frac{c_*}{V_3}$$

Applying Eq. (13a) to the cross-sectional area 3-3 and taking into account the previous relation, it follows that:

$$\left(\frac{V_4}{c_*}\right)^2 = \frac{k-1}{(k+1)\left(1-T_3/T_1\right)} \tag{26}$$

By introducing relations (4c) into Eq. (26), it comes that:

$$\left(\frac{V_4}{c_*}\right)^2 = \frac{k-1}{(k+1)\left[1-(P_3/P_1)^{(k-1)/k}\right]} \tag{26a}$$

And:

$$\left(\frac{V_4}{c_*}\right)^2 = \frac{k-1}{(k+1)\left[1-(\rho_3/\rho_1)^{(k-1)}\right]} \tag{26b}$$

On the other hand, Eq. (5) allows writing that:

$$\rho_4 = \rho_3 V_3 / V_4 \tag{5a}$$

While Eq. (25) gives:

$$V_4 = c_*^2 / V_3 \tag{25a}$$

Introducing Eq.(25a) into Eq.(5a) results in:

$$\rho_4 = \rho_3 V_3^2 / c_*^2 \tag{27}$$

Dividing Eq. (27) by ρ_1 results in:

$$\frac{\rho_4}{\rho_1} = \frac{\rho_3}{\rho_1}\left(\frac{V_3}{c_*}\right)^2 \tag{27a}$$

Taking into account Eqs.(4a) and (13a), Eq.(27a) is reduced to:

$$\frac{\rho_4}{\rho_1} = \left(\frac{T_3}{T_1}\right)^{1/(k-1)} \frac{k+1}{k-1}\left(1-T_3/T_1\right) \tag{28}$$

Combining Eqs. (9), (5a) and (25a), on may write:

$$P_3 - P_4 = \rho_3 (V_3/c_*)^2 (c_*^2/V_3)^2 - \rho_3 V_3^2$$

After some simplifications, the previous relation reduces to:

$$P_3 - P_4 = \rho_3 (c_*^2 - V_3^2) \tag{29}$$

On the other hand, taking into account Eqs. (4c) and (13a), one can deduce that:

$$\left(\frac{V}{c_*}\right)^2 = \frac{k+1}{k-1}\left[1 - \left(\rho / \rho_1\right)^{k-1}\right] \tag{13b}$$

Eq. (29) then becomes:

$$P_3 - P_4 = \rho_3 \left[c_*^2 - \frac{k+1}{k-1} c_*^2 + \frac{k+1}{k-1} c_*^2 \left(\rho_3 / \rho_1\right)^{k-1} \right]$$

Or:

$$P_3 - P_4 = \rho_3 \left[\frac{-2}{k-1} c_*^2 + \frac{k+1}{k-1} c_*^2 \left(\rho_3 / \rho_1\right)^{k-1} \right]$$

Eliminating c_*^2 between Eq.(7) and the previous one, results in:

$$P_3 - P_4 = \rho_3 \left[\frac{-2}{k-1} \frac{2k}{k+1} \frac{P_1}{\rho_1} + \frac{k+1}{k-1} \frac{2k}{k+1} \frac{P_1}{\rho_1} \left(\rho_3 / \rho_1\right)^{k-1} \right]$$

Eliminating ρ_3 between the last equation and equation (4b), it follows that:

$$P_3 - P_4 = \rho_1 \left(\frac{P_3}{P_1}\right)^{1/k} \frac{2k}{k^2-1} \frac{P_1}{\rho_1} \left[-2 + (k+1)\left(\frac{P_3}{P_1}\right)^{(k-1)/k} \right]$$

Whence:

$$\frac{P_4}{P_1} = \frac{P_3}{P_1} - \left(\frac{P_3}{P_1}\right)^{1/k} \frac{2k}{k^2-1} \left[-2 + (k+1)\left(\frac{P_3}{P_1}\right)^{(k-1)/k} \right]$$

After some simplifications and rearrangements, one may obtain the following final result:

$$\frac{P_4}{P_1} = \frac{4k}{k^2-1}\left(\frac{P_3}{P_1}\right)^{1/k} - \frac{k+1}{k-1}\left(\frac{P_3}{P_1}\right) \tag{30a}$$

Eliminating P_3 / P_1 between (30a) and (4a) results in the following equations:

$$\frac{P_4}{P_1} = \frac{4k}{k^2 - 1}\left(\frac{T_3}{T_1}\right)^{1/(k-1)} - \frac{k+1}{k-1}\left(\frac{T_3}{T_1}\right)^{k/(k-1)} \tag{30b}$$

$$\frac{P_4}{P_1} = \frac{4k}{k^2 - 1}\left(\frac{\rho_3}{\rho_1}\right) - \frac{k+1}{k-1}\left(\frac{\rho_3}{\rho_1}\right)^{k} \tag{30c}$$

According to Eq.(1):

$$\rho_4 = \frac{P_4}{gRT_4} = \frac{P_4 / T_1}{gR(T_4 / T_1)} \tag{1}$$

Whence:

$$\frac{\rho_4}{\rho_1} = \frac{P_4 / T_1}{gR\rho_1 (T_4 / T_1)}$$

According to Eq.(1), one may write $P_1 = \rho_1 gRT_1$. Thus, the previous equation becomes:

$$\frac{\rho_4}{\rho_1} = \frac{P_4 / P_1}{(T_4 / T_1)} \Rightarrow T_4 / T_1 = \frac{P_4 / P_1}{\rho_4 / \rho_1}$$

Inserting Eqs. (28) and (30) into the previous equation and taking into account Eqs.(4), the following equations are obtained after some simplification and rearrangements:

$$\frac{T_4}{T_1} = \frac{\dfrac{4k}{(k+1)^2} - \dfrac{T_3}{T_1}}{1 - \dfrac{T_3}{T_1}} \tag{31a}$$

Note some particular values given by Eq. (31a) for $k = 1.4$ such that:

$$T_3 / T_1 = 4k / (k+1)^2 = 0.972222 \Rightarrow T_4 / T_1 = 0,$$

$$T_3 / T_1 = 2 / (k+1) = 0.833333 \Rightarrow T_4 / T_1 = 2 / (k+1) = 0.833333,$$

corresponding to the critical temperature ratio expressed by Eq. (8a).

$$\frac{T_4}{T_1}=\frac{\frac{4k}{(k+1)^2}-\left(\frac{P_3}{P_1}\right)^{(k-1)/k}}{1-\left(\frac{P_3}{P_1}\right)^{(k-1)/k}} \tag{31b}$$

$$\frac{T_4}{T_1}=\frac{\frac{4k}{(k+1)^2}-\left(\frac{\rho_3}{\rho_1}\right)^{k-1}}{1-\left(\frac{\rho_3}{\rho_1}\right)^{k-1}} \tag{31c}$$

DISCUSSION

Parametric formulas (12a), (12b) and (12c) are applicable to all isentropic flows. For this reason, these formulas have the character of general validity and can be applied without distinction to compressible flows, as long as their regime can be considered as isentropic without friction and without pressure loss. The subsonic or supersonic nature of the flow has no influence from the point of view of their applicability.

The physical meaning of the real values of A_*, P_*, ρ_*, T_* and c_* is explained by figure 1 which shows that it is the narrowed section 2-2 of the nozzle which constitutes the geometrical locus of the elements distinguished by the symbol "*". It is further noted that the ratio A/A_* is, in all cases, greater than or equal to 1 and that each time the value of this ratio is greater than 1, there are two sections of the nozzle which correspond to it. One, intersecting the convergent, is the locus of a subsonic flow characterized by the pressure P', the density ρ', the temperature T' and the velocity V'. The other section intersects the divergent and is the locus of a supersonic flow characterized by the pressure P'', the density ρ'', the temperature T'' and the velocity V'' (Figure 1).

These observations are in perfect agreement with relations (12a), (12b) and (12c) each having two real roots determining the values of P', P'', ρ', ρ'', T' et T''. In addition, they define, by applying the continuity equation, the corresponding values of the average velocities V' and V'' in the said sections.

The physical significance of the hypothetical values A_*, P_*, ρ_*, T_*, and c_* is explained by considering figure 6. In the narrowed section of the nozzle, one can write. $P_2/P_1>P_*/P_1$. This implies, by virtue of relations (4) that $T_2/T_1 > T_*/T_1$ and that $\rho_2/\rho_1 > \rho_*/\rho_1$. This allows us to write, taking into account relations (7) and (10) that $V < c_*$. The flow is therefore subsonic.

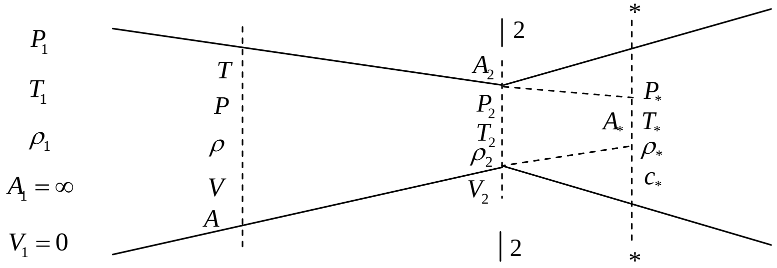


Figure 8: Characteristics of the flow in some sections of the Laval nozzle. * - * Fictitious or hypothetical sonic section obtained by the extension of the convergent

If the convergence of the nozzle continued beyond section 2-2 of Figure 8 and if the decrease in ratios T/T_1, P/P_1 and ρ/ρ_1 continued according to the law governing their variations upstream of section 2-2, these ratios would reach, in a hypothetical cross-sectional area $A = A_*$, the values $P/P_1 = P_*/P_1$, $T/T_1 = T_*/T_1$, $\rho/\rho_1 = \rho_*/\rho_1$ and the velocity V would become sonic, i.e. $V = c_*$.

Applying the continuity equation expressed by relation (5), one can write:

$$A^2 = \left(c_*^2 A_2^2 \rho_2^2\right)/(V^2 \rho^2)$$

By eliminating the quantity V^2/c_*^2 between this last equation and relation (13a), it comes that:

$$\left(\frac{A}{A_2}\right)^2 = \frac{k-1}{k+1}\,\frac{\left(\rho_2/\rho\right)^2}{1-T/T_1}$$

Taking into account relations (8c) and (4c), and the fact that the flow is really sonic in section 2-2, the above relation becomes:

$$\left(\frac{A}{A_*}\right)^2 = \frac{\left(\dfrac{k-1}{k+1}\right)\left(\dfrac{2}{k+1}\right)^{2/(k-1)}}{\left(\dfrac{T}{T_1}\right)^{2/(k-1)} - \left(\dfrac{T}{T_1}\right)^{(k+1)/(k-1)}} \tag{12a}$$

Relation (12a) is thus reproduced. This clearly shows that relations (12a), (12b) and (12c) are valid for any section of the flow slice between sections 1 and 3. This is moreover quite evident from the fact that the flow remains isentropic along the entire length of this slice. For this same reason, the parametric ratios T/T_1, P/P_1 and ρ/ρ_1 follow the isentropic law expressed by equations (4a), (4b) and (4c).

The five parametric relationships that we have established and which express the following ratios:

$$(V/c_*)^2,\ (A/A_*)^2,\ T/T_1,\ P/P_1 \text{ , and } \rho/\rho_1$$

are not independent of each other. It suffices for one of them to be known to deduce the other four.

Regarding the parametric relationships in section 6-6 of figure 5, downstream of the nozzle outlet, they can be determined by applying the energy equation (6) between sections 4-4 and 6 -6. This allows writing, with the aid of Eq.(1), that:

$$V_4^2 + \frac{2k}{k-1} gRT_4 = \frac{2k}{k-1} gRT_6 \tag{32}$$

Whence:

$$T_6 = T_4 + \frac{k-1}{2kgR} V_4^2 \tag{32a}$$

We know that T_4 is given by relation (31a), V_4 by relation (26) and c_* by relation (7). Eq.(32a) then becomes:

$$\frac{T_6}{T_1} = \frac{\dfrac{4k}{(k+1)^2} - \dfrac{T_3}{T_1} + \dfrac{(k-1)^2}{(k+1)^2}}{1 - \dfrac{T_3}{T_1}}$$

After simplifications, the previous equation reduces to:

$$\frac{T_6}{T_1} = \frac{1 - T_3/T_1}{1 - T_3/T_1} = 1 \tag{32b}$$

Meaning that:

$$T_6 = T_1 \tag{32c}$$

To confirm this result, write the energy equation (6) between sections 1-1 and 6-6, with the aid of relation (1). It comes that:

$$\frac{k}{k-1} gRT_1 = \frac{k}{k-1} gRT_6 \tag{6e}$$

It is therefore clear that $T_6 = T_1$.

The pressure parameter in section 6-6 can be expressed as the product of two ratios:

$$\frac{P_6}{P_1} = \frac{P_4}{P_1}\frac{P_6}{P_4} \tag{33}$$

According to Eq. (4a) and taking into account Eq. (32c), one may write:

$$\frac{P_6}{P_4} = \left(\frac{T_6}{T_4}\right)^{k/(k-1)} = \left(\frac{T_1}{T_4}\right)^{k/(k-1)} \tag{4d}$$

However, T_1 / T_4 is expressed by relation (31a) and P_4 / P_1 is expressed by relation (30b). Eq. (33) then becomes:

$$\frac{P_6}{P_1} = \frac{\left[\frac{4k}{(k^2-1)}\left(\frac{T_3}{T_1}\right)^{1/(k-1)} - \frac{k+1}{k-1}\left(\frac{T_3}{T_1}\right)^{k/(k-1)}\right]\left(1-T_3/T_1\right)^{k/(k-1)}}{\left[\frac{4k}{(k+1)^2} - T_3/T_1\right]^{k/(k-1)}} = \psi(T_3/T_1) \tag{34}$$

For a given gas, relation (34) indicates that the ratio P_6 / P_1 is only a function of T_3 / T_1, in the presence of a normal shock wave. For $T_3 / T_1 = 2/(k+1)$, corresponding to the critical temperature ratio, i.e. $T_3 = T_*$, Eq. (34) gives $P_6 / P_1 = \psi(T_3 / T_1) = 1$.

On the other hand, the density parameter in section 6-6 is written, according to relation (1) as:

$$\frac{\rho_6}{\rho_1} = \frac{P_6}{P_1}\frac{T_1}{T_6} \tag{1a}$$

As $T_6 = T_1$, the relation (1a) shows that:

$$\frac{\rho_6}{\rho_1} = \frac{P_6}{P_1} = \psi(T_3/T_1) \tag{35}$$

In the field of the isentropic section located downstream of the shock zone, the pressure parameter in any arbitrarily chosen section is written by taking account of relation (4a) as:

$$\frac{P}{P_6}=\left(\frac{T}{T_6}\right)^{k/(k-1)}=\left(\frac{T}{T_1}\right)^{k/(k-1)}$$, This is allowed since $T_6=T_1$

On the other hand, one may write:

$$\frac{P}{P_1}=\frac{P}{P_6}\frac{P_6}{P_1}$$

Eliminating P/P_6 and P_6/P_1 between the previous equation, (4a) and (34) results in the following functional relationship:

$$\frac{P}{P_1}=\left(\frac{T}{T_1}\right)^{k/(k-1)}\psi\left(T_3/T_1\right) \tag{36}$$

Equation (36) shows that the pressure parameter, which depended on a single variable upstream of the shock (i.e. A/A_*, V/c_*, T/T_1 or ρ/ρ_1), becomes dependent on two variables downstream of the shock. The first variable concerns the state of the gas in the section considered and the other variable concerns the state of the gas in the initial section of the shock.

For $T=T_6$ and knowing that $T_6=T_1$ as stated by Eq. (32c), Eq. (36) is reduced to:

$$\frac{P_6}{P_1}=\psi\left(T_3/T_1\right) \tag{34}$$

Eq. (34) is then reproduced. Equation (36) is only one of the explicit forms of the relation which exists between the three variables P/P_1, T/T_1 and T_3/T_1, the other two are:

$$\frac{T}{T_1}=\left[\frac{P/P_1}{\psi\left(T_3/T_1\right)}\right]^{(k-1)/k} \tag{37}$$

Thus:

$$\psi(T_3/T_1)=\frac{P/P_1}{(T/T_1)^{k/(k-1)}} \tag{38}$$

The parameter of the density can be written as:

$$\frac{\rho}{\rho_1} = \frac{\rho}{\rho_6}\frac{\rho_6}{\rho_1} \tag{39}$$

However, it has been shown that:

$$\frac{\rho_6}{\rho_1} = \frac{P_6}{P_1}$$

Thus, Eq.(39) becomes:

$$\frac{\rho}{\rho_1} = \frac{\rho}{\rho_6}\frac{P_6}{P_1} \tag{39a}$$

Taking into account Eqs. (4b) and (34), one may write the following functional relationship:

$$\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{1/(k-1)} \psi(T_3 / T_1) \tag{40}$$

Conversely, it comes that:

$$\frac{T}{T_1} = \left(\frac{\rho / \rho_1}{\psi(T_3 / T_1)}\right)^{k-1} \tag{41}$$

$$\psi(T_3 / T_1) = \frac{\rho / \rho_1}{\left(T / T_1\right)^{1/(k-1)}} \tag{42}$$

For $\psi(T_3 / T_1) = 1$, corresponding to $T_3 / T_1 = 2/(k+1)$, Eq. (42) gives $\rho_6 / \rho_1 = 1$ since $T_6 / T_1 = 1$.

The energy equation (6) applied between on the one hand any arbitrarily chosen section of the considered field (i.e. downstream of the shock) and section 6-6 on the other hand (Figure 5) gives:

$$V^2 + \frac{2k}{k-1} gRT = \frac{2k}{k-1} gRT_6$$

Taking into account that $T_6 = T_1$ [Eq. (32c)], the previous equation becomes:

$$V^2 = \frac{2k}{k-1} gRT_1 (1 - T / T_1) \tag{10a}$$

Eq.(10a) is then reproduced.

Introducing Eq.(7) results in:

$$\left(\frac{V}{c_*}\right)^2 = \frac{k+1}{k-1}\left(1 - T / T_1\right) \tag{13a}$$

Eq. (13a) is the reproduced.

This shows that equation (13a) which determines the value of the velocity parameter, valid in the upper isentropic domain, retains its validity over the entire length of the flow passing through the nozzle.

On the other hand, applying the continuity equation (5), one may obtain:

$$A^2 = \frac{c_*^2 A_*^2 \rho_*^2}{V^2 \rho^2} \tag{5a}$$

Taking into account Eqs. (13a) and (8c), Eq. (5a) gives:

$$\left(\frac{A}{A_*}\right)^2 = \frac{k-1}{k+1} \frac{\left(\frac{2}{k+1}\right)^{2/(k-1)}}{1 - T / T_1} \left(\frac{\rho_1}{\rho}\right)^2$$

Eliminating ρ / ρ_1 between the previous equation and (40) results in:

$$\left(\frac{A}{A_*}\right)^2 = \frac{k-1}{k+1} \frac{\left(\frac{2}{k+1}\right)^{2/(k-1)}}{1 - T / T_1} \left(\frac{T_1}{T}\right)^{2/(k-1)} \psi^{-2}(T_3 / T_1) \tag{43}$$

Eq. (43) can be written as the product of two functional relationships such that:

$$\left(\frac{A}{A_*}\right)^2 = \varphi_T\left(\frac{T}{T_1}\right)\psi^{-2}(T_3 / T_1) \tag{43a}$$

Where $\varphi_T\left(T/T_1\right)$ is as:

$$\varphi_T\left(T/T_1\right)=\frac{k-1}{k+1}\frac{\left(\frac{2}{k+1}\right)^{2/(k-1)}}{1-T/T_1}\left(\frac{T_1}{T}\right)^{2/(k-1)} \tag{44}$$

By squaring, equation (36) becomes:

$$\left(\frac{P}{P_1}\right)^2=\left(\frac{T}{T_1}\right)^{2k/(k-1)}\psi^2\left(T_3/T_1\right) \tag{36a}$$

The product of equations (36a) and (43a) gives, after deducing the square root:

$$\frac{A}{A_*}\frac{P}{P_1}=\left(\frac{T}{T_1}\right)^{k/(k-1)}\sqrt{\varphi_T\left(\frac{T}{T_1}\right)}=\zeta\left(\frac{T}{T_1}\right) \tag{45}$$

Proceeding in a similar manner by eliminating $\psi(T_3/T_1)$ between equations (40) and (43a), results in:

$$\frac{A}{A_*}\frac{\rho}{\rho_1}=\left(\frac{T}{T_1}\right)^{1/(k-1)}\sqrt{\varphi_T\left(\frac{T}{T_1}\right)}=\xi\left(\frac{T}{T_1}\right) \tag{46}$$

In summary, in any arbitrarily chosen section intersecting the isentropic domain of the flow downstream of the shock zone, the parametric relations are as follows:

$T_6/T_1=1$ (32c);

$P_6/P_1=\rho_6/\rho_1=\psi(T_3/T_1)$ is defined by Eq. (35);

P/P_1 is defined by Eq. (36);

ρ/ρ_1 is defined by Eq. (40);

T/T_1 is defined by Eq. (37) and (41);

$(V/c_*)^2$ is defined by Eq. (13a);

$$(A/A_*)^2=\varphi_T\,(T/T_1)\psi^{-2}(T_3/T_1)$$

is defined by Eq. (43a);

$$(A/A_*)(P/P_1) = \zeta(T/T_1)$$

is defined by Eq. (45) , and

$(A/A_*)(\rho/\rho_1) = \xi(T/T_1)$ is defined by Eq. (46).

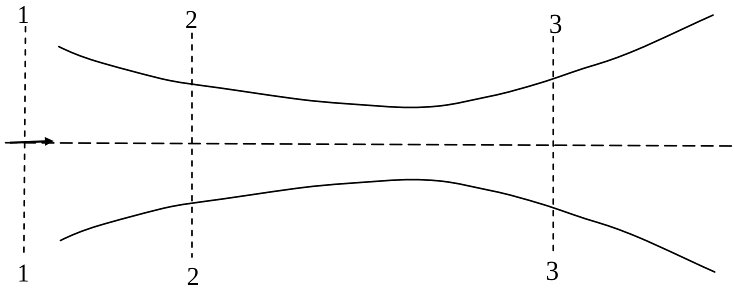
NUMERICAL EXAMPLE 1

A subsonic flow of oxygen, passing through the nozzle shown in the figure below, is characterized by the following parameters:

$A_2 = 8$ cm^2; $P_2 = 9.37\text{x}10^4$ Pa; $T_2 = 294.5$ °K

$A_3 = 5$ cm^2; $P_3 = 8.05\text{x}10^4$ Pa

Determine the mass flow rate.



SOLUTION

The data are insufficient to apply relations (11). One can find a direct solution to this problem. Let us first recall relation (12a):

$$\left(\frac{A}{A_*}\right)^2 = \frac{\left(\frac{k-1}{k+1}\right)\left(\frac{2}{k+1}\right)^{2/(k-1)}}{\left(\frac{T}{T_1}\right)^{2/(k-1)} - \left(\frac{T}{T_1}\right)^{(k+1)/(k-1)}} \qquad (12a)$$

Next, let's write that:

$$\left(\frac{A_2}{A_3}\right)^2 = \left(\frac{A_2 / A_*}{A_3 / A_*}\right)^2 \tag{17}$$

Combining Eqs. (12a) and (17) and after some simplifications, results in:

$$\left(\frac{A_2}{A_3}\right)^2 = \frac{\left(T_3 / T_1\right)^{2/(k-1)} - \left(T_3 / T_{\mathrm{I}}\right)^{\frac{k+1}{k-1}}}{\left(T_2 / T_1\right)^{2/(k-1)} - \left(T_2 / T_1\right)^{\frac{k+1}{k-1}}} \tag{18}$$

The flowing gas being oxygen, its compressibility coefficient is therefore k = 1.4. Inserting this value into Eq.(18) results in:

$$\left(\frac{A_2}{A_3}\right)^2 = \frac{\left(T_3 / T_1\right)^5 - \left(T_3 / T_{\mathrm{I}}\right)^6}{\left(T_2 / T_1\right)^5 - \left(T_2 / T_1\right)^6} \tag{19}$$

By multiplying the numerator and the denominator of the right-hand side of Eq.(19) by the quantity T_1^5, one gets:

$$\left(\frac{A_2}{A_3}\right)^2 = \frac{T_3^{\,5} - T_3^{\,6} T_1^{-1}}{T_2^{\,5} - T_2 T_1^{-1}} \tag{20}$$

Eq.(20) gives T_1 as:

$$T_1 = \frac{\left(A_2 / A_3\right)^2 T_2^{\,6} - T_3^{\,6}}{\left(A_2 / A_3\right)^2 T_2^{\,5} - T_3^5} \tag{21}$$

Eq. (21) has a character of general validity. According to Eq.(4a), one may write:

$$T_3 = T_2 \left(P_3 / P_2\right)^{(k-1)/k} \tag{22}$$

Whence:

$$T_3 = 294.5 \times \left(8.05 \times 10^4 \; / \; 9.37 \times 10^4\right)^{(1.4-1)/1.4} = 282°K$$

With the given values of A_2, A_3, T_2 and T_3, the Eq.(21) gives:

$$T_1 = \frac{(8/5)^2 \times 294.5^6 - 282^6}{(8/5)^2 \times 294.5^5 - 282^5} = 300.234°K \approx 300°K$$

The molar mass of oxygen is M = 32g. According to the relation (2), the constant R is therefore R = 848/32 = 26.5 m/°K.

According to relation (7), the hypothetical sonic velocity is:

$$c_* = \sqrt{\frac{2k}{k+1} gRT_1} = \sqrt{\frac{2 \times 1.4}{1.4+1} \times 9.81 \times 26.5 \times 300} = 301.6417 m/s \approx 302 m/s$$

On the other hand:

$$T_2 / T_1 = 294.5/300 = 0.9816666 \approx 0.982$$

According to Eq.(12a), one may write:

$$\left(\frac{A_2}{A_*}\right)^2 = \frac{\left(\frac{1.4-1}{1.4+1}\right)\left(\frac{2}{1.4+1}\right)^{2/(1.4-1)}}{\left(\frac{T_2}{T_1}\right)^{2/(k-1)} - \left(\frac{T_2}{T_1}\right)^{(1.4+1)/(1.4-1)}}$$

Whence:

$$\left(\frac{A_2}{A_*}\right)^2 = \frac{\left(\frac{1.4-1}{1.4+1}\right)\left(\frac{2}{1.4+1}\right)^{2/(1.4-1)}}{\left(\frac{T_2}{T_1}\right)^{2/(k-1)} - \left(\frac{T_2}{T_1}\right)^{(1.4+1)/(1.4-1)}} = \frac{0.0669796}{\left(\frac{294.5}{300}\right)^5 - \left(\frac{294.5}{300}\right)^6}$$

Hence:

$$\left(\frac{A_2}{A_*}\right)^2 = 4.00756793 \Rightarrow A_* = A_2 / \sqrt{4.00756793} = 8 / \sqrt{4.00756793}$$

The final result is:

$$A_* = 3.99622139 \text{cm}^2 \approx 4 \text{cm}^2$$

According to Eq.(4c), one may write:

$$\left(\frac{T_2}{T_1}\right)^{1/(k-1)} = \frac{\rho_2}{\rho_1} \tag{4c}$$

Thus:

$$\frac{\rho_2}{\rho_1} = \left(\frac{294.5}{300}\right)^{1/(1.4-1)} = 0.95479494 \approx 0.955 \Rightarrow \rho_1 = \rho_2 / 0.955$$

According to Eq.(8c), one may deduce:

$$\rho_* = \left(\frac{2}{k+1}\right)^{1/(k-1)} \rho_1 \tag{8c}$$

Whence:

$$\rho_* = \left(\frac{2}{k+1}\right)^{1/(k-1)} \frac{\rho_2}{0.955}$$

Taking into account Eq.(1), the previous equation is written as

$$\rho_* = \left(\frac{2}{k+1}\right)^{1/(k-1)} \frac{P_2}{0.955 g R T_2}$$

That is:

$$\rho_* = \left(\frac{2}{1.4+1}\right)^{1/(1.4-1)} \frac{9.37\times 10^4}{0.955\times 9.81\times 26.5\times 294.5}$$

Or:

$$\rho_* = \frac{0.63393815\times 9.37\times 10^4}{0.955\times 9.81\times 26.5\times 294.5} = 0.812424\,\text{kg/m}^3$$

The mass flow rate is such that:

$$\dot{m} = c_* \, \rho_* \, A_* = 302\times 0,812424\times 4\times 10^{-4} = 0.0981408\,\text{kg/s}$$

Approximately:

$$\dot{m} = 0.0981\,\text{kg/s}$$

The mass flow rate thus calculated corresponds in fact to the maximum mass flow rate because the example assumes or considers that the flow velocity reaches the sonic velocity in the throat. Thus, the calculated mass flow rate can be found by applying relation (11g).

NUMERICAL EXAMPLE 2

Let us consider the Laval nozzle represented by the figure below, in which the air flow is in the presence of a normal shock wave in the expanded section. The flow regime is subsonic in the convergent, sonic in the narrowed section 2-2.

In the section I-I located in the convergent part, the data is as follows:

$$A_{\mathrm{I}} = \sqrt{2} \times 10^{-2}\, m^2 \;;\; T_{\mathrm{I}} = 287.5°K \;;\; \rho_{\mathrm{I}} = 0.90\, kg/m^3$$

On the other hand, the cross-sectional area 2-2 is as:

$$A_2 = A_* = 10^{-2}\, m^2$$

In section III-III located in the divergent and downstream of the choc zone, the data are as follows:

$$A_{\mathrm{III}} = 2 \times 10^{-2}\, m^2 \;;\; P_{\mathrm{III}} = 77600 Pa$$

Determine the value of the following parameters:

P_{I} ; V_{I}

T_1 ; P_1 ; ρ_1

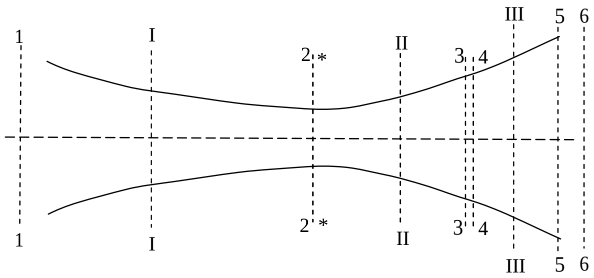
c_* ; T_* ; P_* ; ρ_*

A_3 ; V_3 ; T_3 ; P_3 ; ρ_3

V_4 ; T_4 ; P_4 ; ρ_4

V_{III} ; T_{III} ; P_{III}

P_6 ; ρ_6



SOLUTION

According to relation (1), one can write:

$$P_{\mathrm{I}} = \rho_{\mathrm{I}} g R T_{\mathrm{I}}$$

The molar mass of air is such that M = 28.965338g and the constant *R* is given by equation (2) as:

$$R = \frac{848}{\mathrm{M}} = \frac{848}{28.965338} = 29.276 \approx 29.3 \ m/{}^{\circ}K$$

Thus:

$$P_{\mathrm{I}} = \rho_{\mathrm{I}} g R T_{\mathrm{I}} = 0.90 \times 9.81 \times 29.3 \times 287.5 = 74337.2888 Pa \approx 74373.3 Pa$$

Since the cross-sectional areas A_{I} and A_* are given, then:

$$(A_{\mathrm{I}} / A_*)^2 = \left(\frac{\sqrt{2} \times 10^{-2}}{10^{-2}} \right)^2 = 2$$

For *k* = 1.4, Eq. (12a) becomes:

$$\left(\frac{A}{A_*} \right)^2 = \frac{0.0669796}{\left(\frac{T}{T_1} \right)^5 - \left(\frac{T}{T_1} \right)^6}$$

It is necessary to determine the value of the ratio T_{I} / T_1 of this relation ($A = A_{\mathrm{I}}$ and $T = T_{\mathrm{I}}$) for the known value $\left(A_{\mathrm{I}} / A_* \right)^2 = 2$. The section I-I being located in the convergent where the flow is subsonic, the temperature ratio T_{I} / T_1 should be greater than the critical temperature ratio, i.e. $T_I / T_1 > T_* / T_1 = 5/6 \approx 0.8333$. This is the right branch of the curve in Figure 3 where the flow regime is subsonic. The calculation shows that:

$$T_I / T_1 = 0.958634 \Rightarrow T_1 = T_I / 0.958634 = T_1 = 287.5 / 0.958634$$

$$T_1 = 299.905908°K \approx 300°K$$

For k = 1.4, relation (12b) becomes:

$$\left(\frac{A}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{P}{P_1}\right)^{10/7} - \left(\frac{P}{P_1}\right)^{12/7}}$$

It is necessary to calculate the critical pressure ratio P_I / P_1 from this last relation for the known value $\left(A_I / A_*\right)^2 = 2$. The flow being in subsonic mode, it is about the right branch of figure 3. The P_I / P_1 pressure ratio must be such that:

$P_I / P_1 > P_* / P_1 \approx 0.52828179$. The calculation shows that:

$$P_I / P_1 = 0.86255 \Rightarrow P_1 = 74373.3 / 0.86255 = 86224.9145 Pa$$

$$P_1 \approx 86225 Pa$$

For k = 1.4, Eq. (12c) becomes:

$$\left(\frac{A}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{\rho}{\rho_1}\right)^{2} - \left(\frac{\rho}{\rho_1}\right)^{2.4}}$$

What is needed is to calculate the critical density ratio ρ_I / ρ_1 from this last relation for the known value of $\left(A_I / A_*\right)^2 = 2$. The flow being in subsonic regime, it is about the right branch of figure 4. The ρ_I / ρ_1 density ratio must be such that:

$$\rho_I / \rho_1 > \rho_* / \rho_1 = 0.63393815$$

The calculation shows that:

$$\rho_{\mathrm{I}} / \rho_1 = 0.89977 \Rightarrow \rho_1 = 0.90 / 0.89977 = 1.00025562 kg / m^3$$

$$\rho_1 \approx 1 kg / m^3$$

According to Eq. (13a), one may write for $T = T_{\mathrm{I}}$:

$$\left(\frac{V_{\mathrm{I}}}{c_*}\right)^2 = \frac{k+1}{k-1}(1 - T_{\mathrm{I}} / T_1) = \frac{1.4+1}{1.4-1}(1 - 0.958634) = 0.248196$$

Whence:

$$\frac{V_{\mathrm{I}}}{c_*} = \sqrt{0.248196} = 0.49819273$$

According to Eq. (7) and knowing that R =29.3 and $T_1 = 300°K$, the wave celerity is such that:

$$c_* = \sqrt{\frac{2k}{k+1} gRT_1} = \sqrt{\frac{2 \times 1.4}{1.4+1} \times 9.81 \times 29.3 \times 300} = 317.177474 m / s$$

$$c_* \approx 317.2 m / s$$

Thus:

$$V_{\mathrm{I}} = 0.49819273 c_* = 0.49819273 \times 317.177474 = 158.015512 m / s$$

$$V_{\mathrm{I}} \approx 158 m / s$$

Regarding the parameters in the sonic cross-sectional area namely, T_*, P_* and ρ_*, they can be calculated by applying relations (8) for k = 1.4. It follows:

$$T_* = \frac{2}{k+1} T_1 = \frac{5}{6} T_1 = \frac{5}{6} \times 300 = 250°K$$

$$P_* = \left(\frac{2}{k+1}\right)^{k/(k-1)} P_1 = 0.52828179 \times 86224.9145 = 45551.0522 Pa$$

$$P_* \approx 45551 Pa$$

$$\rho_* = \left(\frac{2}{k+1}\right)^{1/(k-1)} \rho_1 = 0.63393815 \times 1.00025562 = 0.6341002 kg / m^3$$

$$\rho_* \approx 0.634\,kg / m^3$$

Knowing the area of section III-III and the pressure in this section, one can calculate the following parameters:

$$A_{III} / A_* = 2\times10^{-2} / 10^{-2} = 2$$

$$P_{III} / P_1 = 77600 / 86224.9145 = 0.899979$$

$$P_{III} / P_1 \approx 0.90$$

One can also calculate the following product:

$$\frac{A_{III}}{A_*}\frac{P_{III}}{P_1} = 2\times 0.90 = 1.80$$

This product is governed by Eq. (44) for $A = A_{III}$, $P = P_{III}$ and $T = T_{III}$. Thus:

$$\frac{A_{III}}{A_*}\frac{P_{III}}{P_1} = \left(\frac{T_{III}}{T_1}\right)^{k/(k-1)}\sqrt{\varphi\left(\frac{T_{III}}{T_1}\right)} = \zeta\left(\frac{T_{III}}{T_1}\right) = 1.80$$

$$\varphi\left(T_{III} / T_1\right) = \frac{k-1}{k+1}\frac{\left(\frac{2}{k+1}\right)^{2/(k-1)}}{1-T_{III} / T_1}\left(\frac{T_1}{T_{III}}\right)^{2/(k-1)}$$

For k = 1.4, one may write:

$$\frac{A_{III}}{A_*}\frac{P_{III}}{P_1} = \left(\frac{T_{III}}{T_1}\right)^{3.5}\frac{0.25880416}{\left(1-T_{III} / T_1\right)^{1/2}}\left(\frac{T_1}{T_{III}}\right)^{5/2} = 1.80$$

The solution of this equation is:

$$T_{III} / T_1 = 0.98014024$$

Thus:

$$T_{III} = 0.98014024\,T_1 = 0.98014024\times 300 = 294.042072°K$$

$$T_{III} \approx 294.04°K$$

Thus, applying Eq. (1), one may obtain:

$$\rho_{\text{III}} = \frac{P_{\text{III}}}{gRT_{\text{III}}} = \frac{77600}{9.81 \times 29.3 \times 294.04} = 0.91815489\, kg / m^3$$

$$\rho_{\text{III}} \approx 0.918\, kg / m^3$$

On the other hand, one may write for k = 1.4:

$$\varphi\left(T_{\text{III}} / T_1\right) = \frac{0.0669796}{1 - T_{\text{III}} / T_1}\left(\frac{T_1}{T_{\text{III}}}\right)^5 = \frac{0.0669796}{1 - 0.98014024}\left(\frac{1}{0.98014024}\right)^5$$

Whence:

$$\varphi\left(T_{\text{III}} / T_1\right) = 3.72844229$$

Eq. (43a) gives for the cross-sectional area III-III:

$$\psi^{-2}(T_3 / T_1) = \frac{\left(A_{\text{III}} / A_*\right)^2}{\varphi_T\left(T_{\text{III}} / T_1\right)} = \frac{2^2}{3.72844229} = 1.07283409$$

Thus:

$$\psi(T_3 / T_1) = 1 / \sqrt{1.07283409} = 0.96545874$$

For k = 1.4, Eq. (34) gives $\psi(T_3 / T_1)$ as:

$$\psi(T_3 / T_1) = \frac{\left[5.8333333\left(\frac{T_3}{T_1}\right)^{2.5} - 6\left(\frac{T_3}{T_1}\right)^{3.5}\right]\left(1 - T_3 / T_1\right)^{3.5}}{\left[0.97222222 - T_3 / T_1\right]^{3.5}} = 0.96545874$$

The solution of the previous equation is:

$$T_3 / T_1 = 0.72721$$

One can therefore notice that $T_3 / T_1 = 0.72721$ is lower than the critical temperature ratio $T_* / T_1 = 5/6 \approx 0.8333$, which indicates that section 3-3 is located in the supersonic zone of the flow.

Thus:

$$T_3 = 0.72721 T_1 = 0.72721 \times 300 = 218.163 \approx 218°K$$

According to Eq. (13a), one may write for $T = T_3$:

$$\left(\frac{V_3}{c_*}\right)^2 = \frac{k+1}{k-1}(1-T_3/T_1) = \frac{1.4+1}{1.4-1}(1-0.72721) = 1.63674$$

Whence:

$$V_3 = \sqrt{1.63674}\,c_* = \sqrt{1.63674}\times 317.177474 = 405.791445 m/s$$

$$V_3 \approx 405.8\,m/s$$

According to Eq. (4a), one may obtain for *k* = 1.4 what follows:

$$\frac{P_3}{P_1} = \left(\frac{T_3}{T_1}\right)^{\frac{1.4}{1.4-1}} = \left(\frac{218.163}{300}\right)^{3.5} = 0.32795137 \approx 0.328$$

Whence:

$$P_3 = 0.32795137 P_1 = 0.32795137\times 86225 = 28277.6068 Pa$$

$$P_3 \approx 28277.6 Pa$$

$$\frac{P_3}{P_1} = \left(\frac{\rho_3}{\rho_1}\right)^{1.4} \Rightarrow \rho_3 = \left(\frac{P_3}{P_1}\right)^{1/1.4}\rho_1$$

Thus:

$$\rho_3 = \left(\frac{P_3}{P_1}\right)^{1/1.4}\rho_1 = \left(\frac{28277.06068}{86225}\right)^{1/1.4}\times 1 = 0.45096581 kg/m^3$$

$$\rho_3 \approx 0.451 kg/m^3$$

According to Eq. (26) for *k* = 1.4, one obtains:

$$\left(\frac{V_4}{c_*}\right)^2 = \frac{1.4-1}{(1.4+1)\left(1-T_3/T_1\right)} = \frac{1}{6\left(1-T_3/T_1\right)} = \frac{1}{6\times\left(1-0.72721\right)}$$

After calculations, the following result is obtained:

$$V_4 = 0.78164608\,c_* = 0.78164608\times 317.177474 = 247.920528 m/s$$

$$V_4 \approx 248\,m/s$$

Considering Eq. (30b) for k = 1.4 results in:

$$\frac{P_4}{P_1} = 5.83333333\left(\frac{T_3}{T_1}\right)^{2.5} - 6\left(\frac{T_3}{T_1}\right)^{3.5}$$

$$\frac{P_4}{P_1} = 5.83333333 \times 0.72721^{2.5} - 6 \times 0.72721^{3.5} = 0.66296195$$

$$\frac{P_4}{P_1} \approx 0.663$$

$$P_4 = 0.66296195 \times 86224.9145 = 57163.8378 Pa \approx 57164 Pa$$

Applying Eq. (31a) for k = 1.4 yields:

$$\frac{T_4}{T_1} = \frac{0.97222222 - \frac{T_3}{T_1}}{1 - \frac{T_3}{T_1}} = \frac{0.97222222 - 0.72721}{1 - 0.72721} = 0.89817157$$

$$\frac{T_4}{T_1} \approx 0.8982$$

Thus:

$$T_4 = 0.89817157\, T_1 = 0.89817157 \times 300 = 269.451471°K$$

$$T_4 \approx 269.5°K$$

According to Eq. (1), one may write:

$$\rho_4 = \frac{P_4}{gRT_4} = \frac{57163.8378}{9.81 \times 29.3 \times 269.451471} = 0.73808128\, kg / m^3$$

$$\rho_4 \approx 0.738\, kg / m^3$$

Considering Eq. (35) for $\psi(T_3 / T_1) = 0.96545874$ results in:

$$P_6 = P_1\, \psi(T_3 / T_1) = 86224.9145 \times 0.96545874 = 83246.5973 Pa$$

$$P_6 \approx 83246.6 Pa$$

Eq. (1) gives when taking into account that $T_6 = T_1$:

$$\rho_6 = \frac{P_6}{gRT_6} = \frac{83246.5973}{9.81 \times 29.3 \times 300} = 0.96424323\, kg/m^3 \approx 0.964\, kg/m^3$$

Regarding the loss of mechanical energy caused by the normal shock wave, it is obvious that this transformation is irreversible and is accompanied by an increase in entropy. Knowledge of the ratio $P_6/P_1 = \rho_6/\rho_1 = \psi(T_3/T_1)$ makes it possible to calculate the loss of mechanical energy ΔE due to the normal shock wave per unit of mass, transformed into heat. Indeed, the flow between sections 1-1 and 3-3 on the one hand and between sections 4-4 and 6-6 on the other hand is isentropic without friction and therefore without loss of mechanical energy. It follows that all of the mechanical energy expended on the isothermal expansion between sections 1-1 and 6-6 is dissipated by the shock.

Applying relation (6c) between sections 1-1 and 6-6 and knowing that $T_6 = T_1$, one may obtain after some rearrangements:

$$\Delta E = gRT_1 \ln(P_1/P_6) = gRT_1 \ln \psi^{-1}(T_3/T_1) \tag{6e}$$

Therefore, one can form the energy loss parameter $\Delta E/T_1$ such as:

$$\frac{\Delta E}{T_1} = gR \ln(P_1/P_6) = gR \ln \psi^{-1}(T_3/T_1) \tag{6f}$$

In this last equation, the energy loss parameter $\Delta E/T_1$ has the dimension $m^2 s^{-2} {}^\circ K^{-1}$, which implies that ΔE has the dimension $m^2 s^{-2}$, i.e. *J./kg* (Joule per kilogram).

Taking into account the results obtained previously, it comes that:

$$\frac{\Delta E}{T_1} = 9.81 \times 29.3 \times \ln(86224.9145/83246.5973) = 10.1038197\, m^2.s^{-2}.{}^\circ K^{-1}$$

Thus:

$$\Delta E = 10.1038197\, T_1 = 10.1038197 \times 300 = 3031.1459\, m^2/s^2$$

$$\Delta E \approx 3031.15\, m^2/s^2 = 3031.15\, J/kg$$

The yield η of the nozzle is given by the following relation (Ouziaux and Perrier, 2004):

$$\eta = \frac{(P_4 / P_3)^{(k-1)/k} - 1}{(T_4 / T_3) - 1} \tag{47}$$

Whence:

$$\eta = \frac{(P_4 / P_3)^{(k-1)/k} - 1}{(T_4 / T_3) - 1} = \frac{(57163.8378 / 28277.6068)^{(1.4-1)/1.4} - 1}{(269.451471 / 218.163) - 1}$$

The final result is:

$$\eta = 0.94748896 \approx 0.947 = 94.75\%$$

Note that:

Multiplying Eq. (30b) by P_1 / P_3 results in:

$$\frac{P_4}{P_3} = \frac{4k}{k^2 - 1}\left(\frac{T_3}{T_1}\right)^{1/(k-1)}\left(\frac{P_1}{P_3}\right) - \frac{k+1}{k-1}\left(\frac{T_3}{T_1}\right)^{k/(k-1)}\left(\frac{P_1}{P_3}\right) \tag{30d}$$

On the other hand, Eq. (4c) gives:

$$(T_3 / T_1) = \left(P_3 / P_1\right)^{(k-1)/k} \tag{4e}$$

Combining Eqs. (30d) and (4e) results in:

$$\frac{P_4}{P_3} = \frac{4k}{k^2 - 1}\left(\frac{P_1}{P_3}\right)^{(k-1)/k} - \frac{k+1}{k-1} \tag{48}$$

Or, taking into account Eq. (4e), one may write:

$$\frac{P_4}{P_3} = \frac{4k}{k^2 - 1}\left(\frac{T_1}{T_3}\right) - \frac{k+1}{k-1} \tag{49}$$

On the other hand, multiplying Eq. (31a) by T_1 / T_3, yields:

$$\frac{T_4}{T_3} = \frac{\dfrac{4k}{\left(k+1\right)^2}\dfrac{T_1}{T_3} - 1}{1 - \dfrac{T_3}{T_1}} \tag{50}$$

Eliminating P_4/P_3 and T_4/T_3 between Eqs. (47), (49) and (50), gives:

$$\eta = \frac{\left(\frac{4k}{k^2-1}\frac{T_1}{T_3}-\frac{k+1}{k-1}\right)^{(k-1)/k}-1}{\frac{\frac{4k}{(k+1)^2}\frac{T_1}{T_3}-1}{1-T_3/T_1}-1} = \sigma(T_1/T_3) \quad (51)$$

Eq.(51) can be rewritten as:

$$\eta = \frac{\left(\frac{4k}{k^2-1}\frac{T_1}{T_3}-\frac{k+1}{k-1}\right)^{(k-1)/k}-1}{\left[\frac{4k}{(k+1)^2}\frac{T_1}{T_3}-1\right]\frac{T_1/T_3}{T_1/T_3-1}-1} = \sigma(T_1/T_3) \quad (52)$$

For *k* = 1.4, $T_1 = 300°K$ and $T_3 = 218.163°K$, temperature values previously calculated, Eq. (52) gives the same result than that obtained from Eq. (47).

The in-depth study of the relation (52) showed that the yield is 100%, i.e. $\eta = 1$, if $T_3/T_1 = 2/(k+1)$. This corresponds to the critical temperature ratio expressed by the relation (8a) for $T_* = T_3$, also corresponding to the following equalities $P_* = P_3$ and $\rho_* = \rho_3$. The ratio of these parameters is expressed by Eqs. (8b) and (8c) respectively. In addition, the yield η increases as T_3/T_1 increases for a gas of a constant *k* = 1.4.

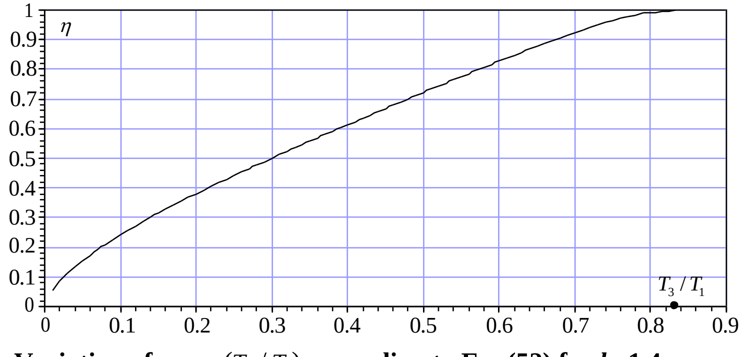


Figure 9: Variation of $\eta = \sigma(T_3/T_1)$ according to Eq. (52) for *k*=1.4.

(•) $T_3/T_1 = 5/6 \approx 0.8333$ **corresponding to** $\eta = 1$

Note

The shock zone can retreat upstream to reach the narrowed section 2-2. This is a borderline case where the cross-sectional areas 2-2, 3-3 and 4-4 coincide or merge. The thickness of the shock zone is reduced to zero. Also, the case where the shock zone moves downstream to reach the final cross-sectional area 5-5 is also a borderline case. In this case, the cross-sectional areas 3-3 and 4-4 coincide or merge with 5-5.

NUMERICAL EXAMPLE 3

A flow of nitrogen, passing through the nozzle shown in the figure below, is defined by the following parameters:

Absolute pressure $P_1 = 10^5 \, Pa$

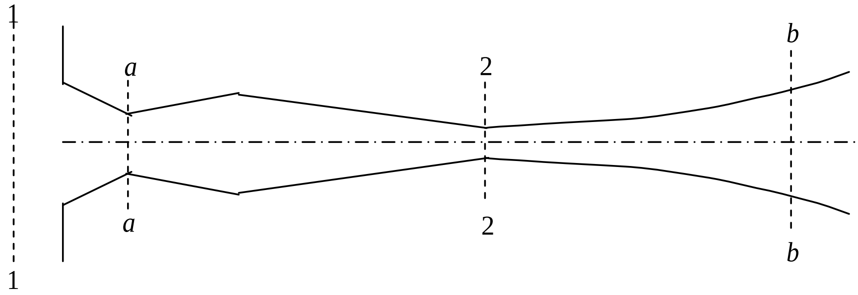
Absolute temperature $T_1 = 293°K$

Mass flow rate $\dot{m} = 1.887 \, kg / s$

Cross-sectional area in section 2-2 $A_2 = 10^{-2} m^2$

Cross-sectional area in section $a-a$ $A_a = \sqrt{2} \times 10^{-2} m^2$

The velocity in the cross-sectional area $b-b$ $V_b = 79.5 \, m / s$



Determine:

1. The velocity V_a, the absolute temperature T_a, the density ρ_a in the cross-sectional area $a-a$.
2. The cross-sectional area A_b.

SOLUTION

The molar mass of nitrogen is M = 28 g.

Thus, according to Eq. (2), the constant R is:

$$R = \frac{848}{\mathrm{M}} = \frac{848}{28} = 30.2857143\, m/°K \approx 30.3\ m/°K$$

According to Eq. (7), one may write:

$$c_* = \sqrt{\frac{2k}{k+1} gRT_1} = \sqrt{\frac{2\times 1.4}{1.4+1}\times 9.81\times 30.3\times 293} = 318.759416\, m/s$$

$$c_* \approx 318.76\, m/s$$

Applying Eq. (1) results in:

$$\rho_{\mathrm{I}} = \frac{P_1}{gRT_{\mathrm{I}}} = \frac{10^5}{9.81\times 30.3\times 253} = 1.3297439\, kg/m^3$$

$$\rho_{\mathrm{I}} \approx 1.33\, kg/m^3$$

According to Relation (8c), the critical value of the density ratio is:

$$\frac{\rho_*}{\rho_1} = \left(\frac{2}{k+1}\right)^{1/(k-1)} = \left(\frac{2}{1.4+1}\right)^{1/(1.4-1)} = 0.63393815$$

Thus:

$$\rho_* = 0.63393815\,\rho_1 = 0.63393815\times 1.3297439 = 0.84297539\, kg/m^3$$

$$\rho_* \approx 0.843\, kg/m^3$$

Taking into account the continuity equation expressed by Eq. (5) results in:

$$A_* = \frac{\overset{*}{m}}{\rho_* c_*} = \frac{1.887}{0.843\times 318.76} = 0.00702232\, m^2 \approx 70.22\times 10^{-4}\, m^2$$

We can thus observe that:

$$A_* = 70.22\times 10^{-4}\, m^2 < A_2 = 10^{-2}\, m^2$$

We can thus conclude that the flow remains subsonic over the entire length of its path.

The section A_*, computed previously, is in fact a hypothetical sonic section. It theoretically occurs at a certain distance downstream from section 2-2 by extending the convergent (Figure 8).

Let us form the following parameter:

$$\left(\frac{A_a}{A_*}\right)^2 = \left(\frac{\sqrt{2}\times 10^{-2}}{0.00702232}\right)^2 = 4.05572949$$

For k = 1.4, Eq. (12a) becomes:

$$\left(\frac{A_a}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{T_a}{T_1}\right)^5 - \left(\frac{T_a}{T_1}\right)^6}$$

It is necessary to determine the value of the ratio T_a / T_1 of this relation ($A = A_a$ and $T = T_a$) for the known value of $\left(A_a / A_*\right)^2 = 4.05572949$. The section $a-a$ being located in the convergent where the flow is subsonic, the temperature ratio T_a / T_1 should be greater than the critical temperature ratio, i.e. $T_a / T_1 > T_* / T_1 = 5/6 \approx 0.8333$. This is the right branch of the curve in Figure 3 where the flow regime is subsonic. The calculation shows that:

$$T_a / T_1 = 0.98190679 \Longrightarrow T_a = 0.98190679 T_1 = 0.98190679 \times 293$$

$$T_a = 287.698689°K \approx 287.7°K$$

According to Eq. (13a), one may write for $T = T_a$:

$$\left(\frac{V_3}{c_*}\right)^2 = \frac{k+1}{k-1}(1 - T_a / T_1) = \frac{1.4+1}{1.4-1}(1 - 0.98190679) = 0.10855926$$

Whence:

$$V_a = \sqrt{0.10855926}\, c_* = \sqrt{0.10855926} \times 318,759416 = 105.025912 m/s$$

$$V_a \approx 105 \, m/s$$

For k = 1.4, relation (12b) becomes:

$$\left(\frac{A_a}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{P_a}{P_1}\right)^{10/7} - \left(\frac{P_a}{P_1}\right)^{12/7}}$$

It is necessary to calculate the critical pressure ratio P_a / P_1 from this last relation for the known value of $\left(A_a / A_*\right)^2 = 4.05572949$. The flow being in subsonic mode, it is about the right branch of figure 3. The P_a / P_1 pressure ratio must be such that:

$P_a / P_1 > P_* / P_1 \approx 0.52828179$. The calculation shows that:

$P_a / P_1 = 0.93809204 \Rightarrow P_a = 0.93809204 \times P_1 = 0.93809204 \times 10^5\,Pa$

$P_a \approx 93810\,Pa$

On the other hand, for k = 1.4, Eq. (12c) becomes:

$$\left(\frac{A_a}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{\rho_a}{\rho_1}\right)^{2} - \left(\frac{\rho_a}{\rho_1}\right)^{2.4}}$$

What is needed is to calculate the critical density ratio ρ_a / ρ_1 from this last relation for the known value of $\left(A_a / A_*\right)^2 = 4.05572949$. The flow being in subsonic regime, it is about the right branch of figure 5. The ρ_a / ρ_1 density ratio must be such that:

$$\rho_a / \rho_1 > \rho_* / \rho_1 = 0.63393815$$

The calculation shows that:

$$\rho_a / \rho_1 = 0.95537819 \Rightarrow$$

$$\rho_a = 0.95537819\rho_1 = 0.95537819 \times 1.3297439 = 1.27040832\,kg / m^3$$

$$\rho_a \approx 1.27\,kg / m^3$$

Let us compute the following ratio:

$$\left(\frac{V_b}{c_*}\right)^2 = \left(\frac{79.5}{318.959416}\right)^2 = 0.06212457$$

According to Eq. (13a), one may write for $T = T_b$:

$$\left(\frac{V_b}{c_*}\right)^2 = \frac{k+1}{k-1}(1 - T_b / T_1) = 0.06212457$$

After some rearrangements, the previous relation gives:

$$T_b / T_1 = 1 - \frac{k-1}{k+1}\left(\frac{V_b}{c_*}\right)^2 = 1 - \frac{1.4-1}{1.4+1} \times 0.06212457 = 0.9896459$$

Applying Eq. (12a) for k = 1.4, one may write:

$$\left(\frac{A_b}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{T_b}{T_1}\right)^5 - \left(\frac{T_b}{T_1}\right)^6} = \frac{0.0669796}{0.9896459^5 - 0.9896459^6} = 6.81444656$$

Whence:

$$A_b = \sqrt{6.81444656} A_* = \sqrt{6.81444656} \times 0.00702232$$

$$A_b = 0.0183341 m^2 \approx 1.83 \times 10^{-2}\ m^2$$

NUMERICAL EXAMPLE 4

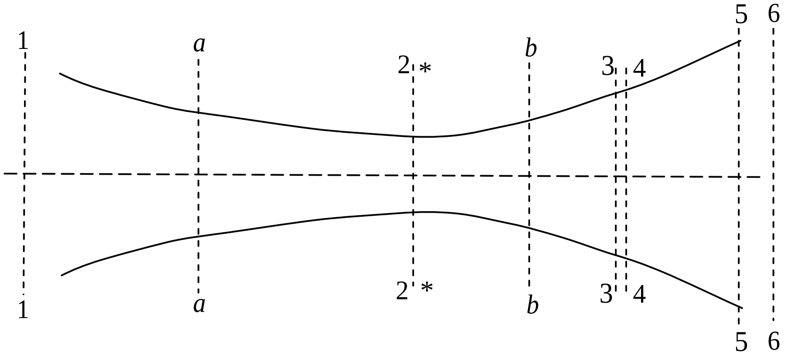
The air flow flowing through the nozzle shown in the figure below is characterized by the following parameters in presence of a zone shock between the sections 3-3 and 4-4, in the expanded section. The flow regime is subsonic in the convergent, sonic in the narrowed section 2-2.

$A_2 = A_* = 20\, cm^2$; $P_1 = 2 \times 10^4\, Pa$; $T_1 = 348.2°K$; $P_6 = 1.22 \times 10^4\, Pa$

Determine:

A_3 (Cross-sectional area 3-3) ; V_3 (Velocity in the section 3-3) ; V_4 (Velocity if the section

4-4) ; $\dot{m}$ (Mass flow rate) ; ΔE (The energy loss)



SOLUTION

The molar mass of air is such that M = 28.965338g and the constant R is given by equation (2) as:

$$R = \frac{848}{M} = \frac{848}{28.965338} = 29.276 \approx 29.3\ m/°K$$

According to Eq. (7), the actual sonic velocity passing through the nozzle is, for k = 1.4:

$$c_* = \sqrt{\frac{2\times 1.4}{1.4+1}\times 9.81\times 29.3\times 348.2} = 341.708744\, m/s \approx 341.71\, m/s$$

Let us calculate the following pressure ratio:

$$P_6 / P_1 = 1.22\times 10^4 / 2\times 10^4 = 0.61$$

Thus, Eq. (34) can be written for k = 1.4 as:

$$\frac{P_6}{P_1} = \frac{\left[5.8333333\left(\frac{T_3}{T_1}\right)^{2.5} - 6\left(\frac{T_3}{T_1}\right)^{3.5}\right]\left(1-T_3/T_1\right)^{3.5}}{\left(0.97222222-T_3/T_1\right)^{3.5}} = 0.61$$

The calculations show that the solution to this equation is:

$$T_3 / T_1 = 0.49911 \approx 0.5$$

Let us recall relation (12a) when applying it for $A = A_3$, $T = T_3$ and $k = 1.4$. Thus:

$$\left(\frac{A_3}{A_*}\right)^2 = \frac{0.0669796}{\left(\frac{T_3}{T_1}\right)^5 - \left(\frac{T_3}{T_1}\right)^6}$$

Whence:

$$\left(\frac{A_3}{A_*}\right)^2 = \frac{0.0669796}{(0.5)^5 - (0.5)^6} = 4.2866941$$

$$A_3 = \sqrt{4.2866941}A_* = \sqrt{4.2866941} \times 20 = 41.4086662\,cm^2$$

$$A_3 \approx 41.41\,cm^2$$

Applying Eq. (13a) to the section 3-3 for $k = 1.4$ results in:

$$\left(\frac{V_3}{c_*}\right)^2 = 6(1 - T_3 / T_1) = 6 \times (1 - 0.5) = 3$$

Thus:

$$V_3 = \sqrt{3}\,c_* = \sqrt{3} \times 341.708744 = 591.856906\,m/s \approx 592\,m/s$$

In the same way, let's apply Eq. (26) to the section 4-4 for $k = 1.4$. So:

$$\left(\frac{V_4}{c_*}\right)^2 = \frac{1}{6\left(1 - T_3 / T_1\right)} = \frac{1}{6 \times \left(1 - 0.5\right)} = 1/3$$

The velocity V_4 is then:

$$V_4 = c_* / \sqrt{3} = 341.708744 / \sqrt{3} = 197.285635\,m/s \approx 197.3\,m/s$$

To compute the mass flow rate, let us apply Eq. (11a) for:

$$A = A_* = A_2 \,;\; T = T_* \,;\; k = 1.4$$

Thus:

$$\dot{m} = A_2 \rho_1 \left(\frac{T_*}{T_1}\right)^{2.5} \sqrt{7gRT_1\left(1 - T_* / T_1\right)}$$

Combining the previous relation with Eqs. (1) and (8a), one may obtain the following result:

$$\dot{m} = \frac{0.68473145 A_2 P_1}{\sqrt{gRT_1}}$$

Thus:

$$\dot{m} = \frac{0.68473145 \times 20 \times 10^{-4} \times 2 \times 10^{4}}{\sqrt{9.81 \times 29.3 \times 348.2}} = 0.08657601\, kg/s$$

$$\dot{m} \approx 0.0866\, kg/s$$

The mass flow rate thus calculated corresponds in fact to the maximum mass flow rate because the example assumes or considers that the flow velocity reaches the sonic velocity in the throat. Thus, the calculated mass flow rate can be found by applying relation (11g).

Concerning the energy loss, applying relation (6c) between sections 1-1 and 6-6 and knowing that $T_6 = T_1$, one may obtain after some rearrangements:

$$\Delta E = gRT_1 \ln(P_1 / P_6)$$

Whence:

$$\Delta E = 9.81 \times 29.3 \times 348.2 \times \ln(1/0.61) = 49471.2374\, m^2/s^2$$

$$\Delta E \approx 49471.24\, m^2/s^2 \text{ or } \Delta E \approx 49471.24\, J/kg$$

BORDERLINE CASES

1. The shock zone retreats upstream and reaches section 2-2 corresponding to the throttle of the nozzle (Figure 10). As a result, sections 2, 3 and 4 coincide. In this case, one can write:

$$T_4 = T_3 = T_2 \Rightarrow T_3 / T_1 = 2/(k+1) \Rightarrow P_6 / P_1 = \rho_6 / \rho_1 = 1$$

According to Eq. (12a), we have in the terminal section:

$$\left(\frac{A_5}{A_*}\right)^2 = \varphi_T \left(\frac{T_5'}{T_1}\right)$$

where T_5'/T_1 is the temperature ratio corresponding to this borderline case. Having the value of the section ratio previously defined, the value of the temperature ratio is then worked out. This corresponds to the subsonic branch of the curve in Figure 3. Knowing the value of the temperature ratio $T_5^{'} / T_1$, one can deduce the value of $(V_5^{'} / c_*)^2$, $P_5^{'} / P_1$, and $\rho_5^{'} / \rho_1$, corresponding to the current borderline case, by applying the corresponding equation determined above.

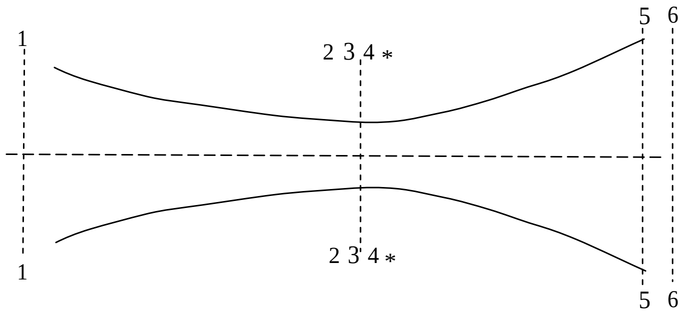


Figure 10: Limit case where the shock zone reaches the narrowed section 2-2 of the nozzle.

2. In a second borderline case, the shock zone moves downstream and reaches terminal section 5 (Figure 11).

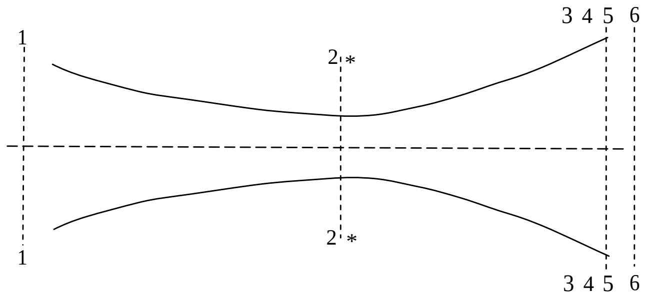


Figure 11: Limit case where the shock zone reaches the terminal section 5-5 of the nozzle.

Consequently, the shock zone, delimited by sections 3 and 4, very close to each other, coincides with the terminal section 5:

- Immediately upstream of the shock, we have the following parameters:

$$\frac{T_{5-3}^{''}}{T_1} = \frac{T_3}{T_1} \; ; \; \frac{P_{5-3}^{''}}{P_1} = \frac{P_3}{P_1} \; ; \; \frac{\rho_{5-3}^{''}}{\rho_1} = \frac{\rho_3}{\rho_1} \text{ , and } \left(\frac{V_{5-3}^{''}}{c_*}\right)^2 = \left(\frac{V_3}{c_*}\right)^2$$

- Immediately after the shock, we have the following parameters:

$$\frac{T''_{5-4}}{T_1} = \frac{T_4}{T_1}\ ;\ \frac{P''_{5-4}}{P_1} = \frac{P_4}{P_1}\ ;\ \frac{\rho''_{5-4}}{\rho_1} = \frac{\rho_4}{\rho_1}\text{, and } \left(\frac{V''_{5-4}}{c_*}\right)^2 = \left(\frac{V_4}{c_*}\right)^2$$

Applying Eq. (12a), one may write:

$$\left(\frac{A_5}{A_*}\right)^2 = \varphi_T\left(\frac{T''_{5-3}}{T_1}\right)$$

This corresponds to the subsonic branch of the curve in Figure 3. Knowing the cross-sectional area ratio A_5 / A_*, the temperature ratio T''_{5-3} / T_1, corresponding to this borderline case, is worked out from Eq. (12a). The value of the last temperature ratio allows calculating the following parameters by applying the corresponding equation determined previously, such as Eqs. (30):

$$\frac{T''_{5-4}}{T_1}\ ;\ \left(\frac{V''_{5-3}}{c_*}\right);\ \left(\frac{V''_{5-4}}{c_*}\right);\ \frac{P''_{5-3}}{P_1}\ ;\ \frac{P''_{5-4}}{P_1}\ ;\ \frac{\rho''_{5-3}}{\rho_1}\text{, and } \frac{\rho''_{5-4}}{\rho_1}$$

CLASSIFICATION OF FLOWS

It is the study of borderline cases that makes it possible to classify the flows studied in this paper into four groups:

1. In the case where:

$$T_5 / T_1 > T'_5 / T_1 \Rightarrow P_5 / P_1 > P'_5 / P_1 \Rightarrow \rho_5 / \rho_1 > \rho'_5 / \rho_1$$

The flow remains subsonic and entropic over the entire length of the nozzle and it is the introduction of the hypothetical values of the parameters c_* and A_* which makes it possible to analyze the problem, taking into account upstream and downstream conditions of the flow in the nozzle.

2. In the case where:

$$T_5 / T_1 = T'_5 / T_1 \Rightarrow P_5 / P_1 = P'_5 / P_1 \Rightarrow \rho_5 / \rho_1 = \rho'_5 / \rho_1$$

We are in the presence of the first borderline case. The flow becomes sonic only over the infinitesimal distance corresponding to the passage through the throat. There is no shock wave and the flow is isentropic over the entire length of the nozzle.

3. In the case where:

$$\frac{T_5^{'}}{T_1} > \frac{T_5}{T_1} > \frac{T_{5-4}^{''}}{T_1} \Rightarrow \frac{P_5^{'}}{P_1} > \frac{P_5}{P_1} > \frac{P_{5-4}^{''}}{P_1}\text{, and } \frac{\rho_5^{'}}{\rho_1} > \frac{\rho_5}{T_1} > \frac{\rho_{5-4}^{''}}{\rho_1}$$

The flow, subsonic upstream of the narrowed section, becomes sonic in the latter to immediately transform into supersonic immediately downstream of this same section. This supersonic flow accelerates until it reaches section 3, to change there again in subsonic, producing a normal shock wave and decelerating until the exit of the nozzle. It is this general case which has been the subject of detailed considerations in this paper. The characteristics of such a flow upstream of the shock zone only depend on the upstream conditions, while the position of the shock zone and the characteristics of the section downstream thereof are influenced by the conditions existing downstream of the nozzle.

4. In the case where:

$$\frac{T_{5-4}^{''}}{T_1} \geq \frac{T_5}{T_1} \Rightarrow \frac{P_{5-4}^{''}}{P_1} \geq \frac{P_5}{P_1} \Rightarrow \frac{\rho_{5-4}^{''}}{\rho_1} \geq \frac{\rho_5}{\rho_1}$$

The flow, subsonic upstream of the narrowed section, becomes sonic there to transform into supersonic immediately downstream of this section. The flow will retain this nature of its regime until the outlet of the nozzle. Such a flow remains isentropic over the entire length of its path in the nozzle and depends only on the conditions existing upstream.

Note

To determine all the characteristics of the isentropic flow upstream of the shock, or if there is none along the entire length of the nozzle, it is sufficient to know the groups G_1 made up of the following elements:

$$A,T,P \,;\; A,T,\rho \,;\; A,P,\rho \,;\; A,V,\rho \,;\; A,V,P \quad (G_1)$$

of which A and V in the section *-*, the others T, P or ρ either in the section *-*, either in the upstream compartment of the nozzle, or in a section which is arbitrarily chosen provided one of the following five parameters must then be known:

$$\left(\frac{A}{A_*}\right)^2 ; \left(\frac{V}{c_*}\right)^2 ; \frac{T}{T_1} ; \frac{P}{P_1}\text{, and } \frac{\rho}{\rho_1} \quad (G_2)$$

When the flow is complicated by the presence of a shock wave, the complete determination of all its characteristics requires knowledge, in a section downstream of the shock, of one of the following pairs of parameters:

$$\frac{A}{A_*},\frac{P}{P_1};\ \frac{A}{A_*},\frac{\rho}{\rho_1};\ \frac{A}{A_*},\frac{T}{T_1};\ \frac{A}{A_*},\frac{V}{c_*};\ \frac{T}{T_1},\frac{P}{P_1};\ \frac{T}{T_1},\frac{\rho}{\rho_1};\ \frac{P}{P_1},\frac{\rho}{\rho_1};\ \frac{V}{c_*},\frac{\rho}{\rho_1};\ \frac{V}{c_*},\frac{P}{P_1};\ \frac{V}{c_*},\frac{\rho}{\rho_1}$$

and one of the G_1 group in the same section including A and V in the *-* section, the other T, P and ρ either in the *-* section, or in the upstream compartment of the nozzle, or in a arbitrarily chosen section. In this case, one must know one of the five parameters of G_2 group.

CONCLUSIONS

The Laval nozzle has been the subject of a theoretical development in order to define the functional relationships which link the flow parameters between them. The parameters of the flow in any cross-sectional of the nozzle have all been related to the parameters of the flow at the generator state, with the exception of the cross-sectional area A which has been related to the narrowed section A_* of the nozzle. The parameters thereof are characterized by the index "1". On the other hand, the flow parameters in the narrowed section of the nozzle corresponding to the Mach number M_* are endowed with the index "*".

The energy equation allowed determining the relationships governing the velocity ratio V/V_*, where V is the average velocity of the isentropic flow, subsonic or supersonic, in any cross-sectional area of the nozzle [Eqs. (10a), (10b) and 10c)].

The velocity equations, the continuity equation as well as the relations governing the isentropic flow made it possible to define the relationships of the mass flow rate. This is given as a function of the flow parameters related to those of the flow in the generator state such as T/T_1, P/P_1 and ρ/ρ_1 [Eqs.(11a), (11b) and (11c)].

The energy equation, the continuity equation as well as the isentropic relations made it possible to determine the relationships governing the cross-sectional area ratio A/A_*, where A is any cross-sectional area [Eqs. (12a), (12b) and 12c)]. The quadratic form of the obtained equations made it possible to conclude that there are two cross-sectional areas A characterized by two different values of the flow parameters such as the temperature ratios T/T_1, the pressure ratio P/P_1 and density ratio ρ/ρ_1. The functional relationships $/A_* = f\,(T/T_1)$, $A/A_* = f\,(P/P_1)$ and $A/A_* = f\,(\rho/\rho_1)$ have been illustrated in figures 3, 4 and 5.

The Theoretical development could lead to the establishment of the relation between the Mach number M_* in the narrowed section A_* of the nozzle as a function of the Mach number M in any section A. The graphical representation of the relation [Fig (7)] showed that the flow is sonic in the throat if and only if the Mach number M is equal to unity. For values of M such as $0 < M < 1$, corresponding to a subsonic flow, the throat is also the geometrical locus of a subsonic flow. However, for this range of values of the Mach number M, the sonic character of the flow may occur in a hypothetical cross-sectional area A_* located downstream of the throat at some distance from it.

The normal shock zone has been the subject of theoretical development in order to define the relations which bind the parameters of the flow. Based on two very close sections 3-3 and 4-4 (Figure 6), the theoretical development led to defining the following functional relationships:

$$(A_4 / A_*)^2 = \varphi(T_3 / T_1) = \xi(P_3 / P_1) = \zeta(\rho_3 / \rho_1).$$

Also, the following ratios have been determined:

$$(V_4 / c_*)^2,\ T_4 / T_1,\ \rho_4 / \rho_1, \text{ and } P_4 / P_1.$$

Finally, the borderline cases were examined and made it possible to define the criteria for classifying flows in the nozzle.

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