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NEW THEORETICAL CONSIDERATIONS ON THE FLOW PARAMETERS IN THE TRANSITION AND SMOOTH REGIMES

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ABSTRACT

Transition and smooth flows are often encountered during experiments in the laboratory and even sometimes in the field. The transition domain occupies a fairly large space in Moody's diagram while the smooth flow is reduced to a curve which represents the lower envelope of the diagram. The characteristic length corresponding to these two domains, such as the width of a channel or the flow depth, is currently calculated by an iterative process such as the trial and error method. To overcome this drawback, the present study presents a direct method consisting of first calculating the characteristic length in the domain assumed to be rough. The characteristic length sought is equal to this length corrected for effects of a dimensionless correction coefficient. In the transition domain, the correction coefficient depends both on the Reynolds number and on the relative roughness corresponding to the rough domain while for the smooth regime the correction coefficient depends only on the Reynolds number in the rough zone. Expressions for Reynolds numbers in the transition and smooth domains are also presented. The governing relationships are practical and differ from those usually found in the literature. Practical numerical examples are provided to show both how the method should be applied and the evidence for its reliability.

Keywords: Characteristic length, transition domain, smooth regime, correction coefficient, friction factor.

INTRDODUCTION

In 1944, Moody plotted his famous dimensionless diagram which still bears his name today. This diagram represents the variation of the Darcy-Weisbach friction factor f versus the Reynolds number R for various values of the relative roughness ε / D, where ε is the absolute roughness characterizing the state of the inner wall of a pipe of diameter D.

This arduous labour is an adaptation of the work of Rouse (1943) using Pigott's results (1933), whose work was based on an analysis of about 10,000 experiments from various sources like the one providing by Kemler (1933) and the contribution of Nikuradse (1933).

Moody's main purpose was to provide a graphical representation of the well known implicit Colebrook's relationship (1939) which is a function of three dimensionless parameters namely f, R, and ε / D. This graphic representation resulted in a series of curves in a bi-logarithmic coordinate system where f is represented on the ordinate and R on the abscissa. Each of the curves was plotted for a given value of ε / D.

For a long time and despite the lack of accuracy, this diagram was mainly used to read the value of the friction factor f provided the pair of parameters ε / D and R was given. Knowing the value of f, one could then calculate the pressure drop in a pipe or the flow rate using the Darcy-Weisbach relationship (1854). Later, the graphical reading of f was abandoned in favour of a more precise calculation performed on the implicit Colebrook's relationship thanks to the advent of powerful calculators. Today, one prefers the use of approximate formulas provided by abundant literature and whose precision is excellent, sufficient to solve practical hydraulic engineering problems (Zeghadnia et al., 2019).

Moody's diagram is therefore made up of a series of curves, the lower envelope of which represents the smooth flow regime which corresponds to a relative roughness tending towards zero. At the far right of the diagram, the curves of the diagram are almost horizontal indicating that the variation of f is not influenced by the variation of R. The coefficient f depends only on the relative roughness ε / D. This zone of the diagram corresponds to a rough turbulent flow where the turbulence is complete (Chow, 1959). Between the smooth and rough zones, there is the so-called transition zone where the coefficient f decreases with the increase in R, whatever the relative roughness ε / D. One can separate the two transition and rough zones by a curve. According to Hager (1987), this separating curve is a hyperbola of equation $R\times(\varepsilon / D) = 1050$ derived with a relative error of 1.5%. This amounts to saying that if the pair of values of the parameters R and ε / D is such that $R\times(\varepsilon / D) \geq 1050$, then the flow can be considered, with an excellent approximation, as belonging to the rough turbulent domain.

In the Moody diagram, the relative roughness ε / D have been varied between 0 and 0.05 and the Reynolds number varies from 2300 until reaching the value 10^8.

From a qualitative point of view, the Moody diagram has always been used to identify the nature of the flow regime in a pipe, by placing the pair of the parameters R and ε / D values on the diagram. The point of intersection is located on one of the three areas previously described, which indicates the nature of the flow regime.

On the other hand, although the diagram has been plotted for flow in a full pipe, it is accepted today its extrapolation to flow in open channels (Sinniger and Hager, 1989). In this case, the diameter D is replaced by the hydraulic diameter D_h.

The rough turbulent flow occupies a prominent space in Moody's diagram, which suggests a high probability that the rough turbulent regime will be encountered, especially in practice. In a recent study, the authors were interested in the hydraulic parameters of this flow regime in open channels such as characteristic length, mean flow velocity, Reynolds number, and hydraulic diameter. The authors equated these parameters through new theoretical considerations based on the combination of both Darcy-Weisbach (1854) and Nikuradse (1933) rational relationships. Emphasis has been placed on the characteristic length L_R which can be either the linear dimension of a channel such as its width b or the flow depth h…etc, the subscript "*R*" denotes "*Rough*". Currently, this length is calculated using the tedious trial and error method when the Darcy-Weisbach relationship is used. Furthermore, the use of Manning-Strickler type relationships are not suitable since the resistance coefficient related to these formulas depends on the characteristic length or the depth sought (Achour and Bedjaoui, 2006; Achour and Amara, 2020). Achour and Amara (year) showed that the characteristic length L_R can be expressed as the product of two explicit functions. The first function, which has the dimension of a length, depends on the flow rate Q, the absolute roughness ε and the channel bed slope S_0, all are parameters known in practice. The second function, which is dimensionless, depends exclusively on the aspect ratio of the wetted area. It is worth noting that the L_R relationship is an approximate equation, but the numerous numerical examples carried out have clearly shown its reliability since L_R can be calculated with a relative error of less than 1%. Moreover, the authors recommended the use of the Lambert function (Boyd, 1998) which provides a lower relative error.

What about the characteristic length in both transition and smooth regimes which also occupy a privileged area in the Moody diagram? This is the question that this study should answer. The characteristic length in the transition domain is expected to be equal to the characteristic length in the rough domain corrected for effects of a correction coefficient. This should depend on both Reynolds number and the relative roughness. Likewise, it is expected that the characteristic length in the smooth regime is equal to the characteristic length in the rough domain corrected for effects of a correction factor depending exclusively on the Reynolds number.

AVAILABLE RELATIONSHIPS

Let's define any parameter of the flow in the rough turbulent zone by the subscript "*R*" denoting "*Rough*" and by the subscript "*T*" denoting "*Transition*" any parameter of the flow in the transition domain.

The friction factor f_T is governed by the well known Colebrook relationship (1939) as:

$$f_T^{-1/2} = -2\log\left(\frac{\varepsilon_T^*}{3.7} + \frac{2.51}{R_T\sqrt{f_T}}\right) \tag{1}$$

where the relative roughness ε_T^* is as:

$$\varepsilon_T^* = \frac{\varepsilon}{D_{h,T}} \tag{2}$$

$D_{h,T}$ is the hydraulic diameter in the transition zone, ε is the absolute roughness, R_T is the Reynolds number characterizing the flow in the transition zone.

The friction factor f_R in the rough turbulent domain can be deduced from Eq. (1) writing $R_T \to \infty$. Whence:

$$f_R^{-1/2} = -2\log\left(\frac{\varepsilon_R^*}{3.7}\right) \tag{3}$$

Where the relative roughness ε_R^* is expressed as:

$$\varepsilon_R^* = \frac{\varepsilon}{D_{h,R}} \tag{4}$$

$D_{h,R}$ is the hydraulic diameter in the rough turbulent zone which can be expressed according to the following relation:

$$D_{h,R} = 4L_R\,\frac{A_1}{P_1} \tag{5}$$

where L_R (*m*) is the characteristic length in the rough domain, A_1 and P_1 are the water area and the wetted perimeter respectively for the characteristic length L_R equal to one meter. The dimensionless parameters A_1 and P_1 only depend on the aspect ratio of the wetted area and the characteristic length L_R can be the width *b* of a rectangular channel, or the diameter *D* of a fully or partially filled circular pipe, or even the flow depth *h*... etc, in the rough turbulent zone. Table 1 in the appendix groups together some formulae of $A_1$and $P_1$for various channel shapes in accordance with the chosen characteristic length *L*.

Knowing that the wetted perimeter in the rough domain is as $P_R = L_R P_1$, where P_1 is the wetted perimeter for $L_R = 1$ *m*, the Reynolds number R_R is thus governed by the following relationship:

$$R_R = \frac{4Q}{L_R P_1 \nu} \tag{6}$$

where Q (m^3/s) is the discharge, and ν (m^2/s) is the kinematic viscosity.

In the rough turbulent domain, the relationship resulting from the combination of the well-known formulas of Darcy-Weisbach and Nikuradse governing the rough friction factor can easily be written in the following dimensionless form

$$\varphi = 4\sqrt{2}\beta^{5/2}\log(14.8\beta) \tag{7}$$

where:

$$\varphi = \frac{A_1 Q}{P_1^2 \sqrt{gS_0}\,\varepsilon^{5/2}} \tag{8}$$

and β is as:

$$\beta = \frac{L_R A_1}{\varepsilon P_1} \tag{9}$$

Eq. (7) is the implicit relationship that gives the exact value of the characteristic length L_R. The known parameter is the function φ, what is sought is the β parameter in order to deduce L_R in accordance with Eq. (9). Although Eq. (7) is transcendental which can be solved by trial and error process, an exact analytical method for tackling the implicit Eq. (7) is possible using the Lambert W-function. For that, let us express Eq. (7) as follows:

$$\varphi = \frac{4\sqrt{2}}{\ln(10)}\beta^{5/2}\ln(14.8\beta) \tag{10}$$

which can be written in compact form as:

$$\beta^{5/2}\ln(14.8\beta) = C \tag{11}$$

where:

$$C = \frac{\varphi\ln(10)}{4\sqrt{2}} \tag{12}$$

The exact analytical solution of the transcendental Eq. (11) can be formulated in terms of the Lambert W function. This solution for $C > 0$ reads then:

$$\beta = \frac{5}{74}\exp\left(\frac{2}{5}\mathbf{W}_0\left(\frac{2738\sqrt{370}}{25}\ C\right)\right) \tag{13}$$

In which $\mathbf{W}_0$ is principal branch of the Lambert Function (Corless et al. 1996) which can be easily computed by software package like Maple. Being a transcendental function, formal solution of the Lambert W-function can be expressed only in endless form. However, a perusal of Eqs. (8) and (12) indicates that the argument x involved in $\mathbf{W}_0(x)$ is very large. For $x \to \infty$, the following three-term asymptotic development holds (Boyd, 1998):

$$\mathbf{W}_0(x) = \ln(x) - \ln\ln(x) + \frac{\ln\ln(x)}{\ln(x)} \tag{14}$$

Using Eq.(14), the computation of the Lambert function is greatly simplified. When substituted in Eq. (13), the exact solution for β and hence for the characteristic length L_R is worked out from Eq. (9).

CHARACTERISTIC LENGTH IN THE TRANSITION ZONE

To compute the characteristic length L_T when the current flow prevails in the transition zone, one may assume as a first approximation that the flow is in the rough turbulent domain carried by a hypothetical channel with L_R as the characteristic length. Both the actual and hypothetical channels have the same conveyance $Q/(S_0)^{1/2}$, where S_0 is the channel bed slope. However, L_T is obviously greater than L_R, since the friction loss is more important in the hypothetical channel. Thus, the following relationship between L_T and L_R can be formulated:

$$L_T = \lambda L_R \tag{15}$$

where λ can be defined as the dimensionless correction factor of the characteristic length in the transition zone. Furthermore, λ is greater than unity in the transition domain, while it is equal to unity when the flow is in the rough turbulent zone implying $L_T = L_R$ according to Eq. (15).

REYNOLDS NUMBER IN THE TRANSITION ZONE

When the flow is in the transition zone, Reynolds number R_T can be expressed as:

$$R_T = \frac{4Q}{P_T \nu} \tag{16}$$

where P_T is the wetted perimeter in the transition zone. Eq. (16) is similar to Eq. (6) and can therefore be written as:

$$R_T = \frac{4Q}{L_T P_1 \nu} \tag{17}$$

On the other hand, inserting Eq. (15) into Eq. (17) results in:

$$R_T = \frac{4Q}{\lambda L_R P_1 \nu} \tag{18}$$

Comparing Eqs. (6) and (18) yields:

$$R_T = \lambda^{-1} R_R \tag{19}$$

DIMENSIONLESS CORRECTION FACTOR λ

The wetted area A_T in the transition zone can be written as:

$$A_T = L_T^2 A_1 \tag{20}$$

Applying a correlation statistical analysis on Eq. (7), one may deduce that any characteristic length L_R in the rough turbulent zone is governed by the following relationship:

$$L_R = \Lambda L^* \tag{21}$$

where:

$$\Lambda = \left(\frac{Q \varepsilon^{0.15}}{8.85 \sqrt{g S_0}} \right)^{1/2.65} \tag{22}$$

$$L^* = \frac{P_1^{0.245}}{A_1^{0.623}} \tag{23}$$

Inserting Eqs. (15) and (21) into Eq. (20) results in:

$$A_T = \lambda^2 \Lambda^2 L^{*2} A_1 \tag{24}$$

Eq. (24) can be rewritten as:

$$A_T = \lambda^2 \Lambda^2 A^* \tag{25}$$

where:

$$A^* = L^{*2} A_1 \tag{26}$$

According to Eqs. (15) and (21), the hydraulic diameter $D_{h,T}$ can be written as:

$$D_{h,T} = \lambda \Lambda D_h^* \tag{27}$$

Inserting Eq. (27) into the Darcy-Weisbach relationship, it is easy to obtain what follows:

$$\lambda\Lambda = f_T^{1/5} \left(\frac{Q^2}{2g\, A^{*2} D_h^* S_0} \right)^{1/5} \tag{28}$$

When the flow is in the rough turbulent zone, corresponding to $\lambda = 1$ and $f_T = f_R$, Eq. (28) becomes:

$$\Lambda = f_R^{1/5} \left(\frac{Q^2}{2g\, A^{*2} D_h^* S_0} \right)^{1/5} \tag{29}$$

Inserting Eq. (29) into Eq. (28) and simplifying results in:

$$\lambda = \left(\frac{f_T}{f_R} \right)^{1/5} \tag{30}$$

It is thus demonstrated that the correction factor λ is the ratio of the friction factors f_T and f_R to the power one fifth.

Eq. (30) gives:

$$\sqrt{f_T} = \lambda^{5/2} \sqrt{f_R} \tag{31}$$

On the other hand, according to Eq. (15), Eq. (2) can be rewritten as:

$$\varepsilon_T^* = \frac{\varepsilon}{D_{h,T}} = \frac{\varepsilon}{\lambda\ D_{h,R}} = \lambda^{-1} \varepsilon_R^* \tag{32}$$

Inserting Eqs. (19), (31), and (32) into Eq. (1) yields:

$$\lambda^{-5/2} f_R^{-1/2} = -2\log\left(\frac{\varepsilon_R^*}{3.7\,\lambda} + \frac{2.51}{\lambda^{-1}\lambda^{5/2} R_R \sqrt{f_R}} \right) \tag{33}$$

After some arrangements, Eq. (33) reduces to:

$$\lambda = \left[-2\sqrt{f_R}\,\log\left(\frac{\varepsilon_R^*}{3.7\,\lambda} + \frac{2.51}{\lambda^{3/2} R_R \sqrt{f_R}} \right) \right]^{-2/5} \tag{34}$$

All the parameters of Eq. (34) are known except for the correction factor λ. The friction factor f_R is governed by Eq. (3), ε_R^* is given by Eq. (4) along with Eq. (5), and R_R is explicitly computed using Eq. (6).

The implicit Eq. (34) needs a trial-and-error procedure to compute λ but, as one can see below, this difficulty can be overcome by adopting an approximate method.

Let's adopt the following iterative process applying on the implicit Eq. (34):

$$\lambda_{(1)} = \left[-2\sqrt{f_R}\,\log\left(\frac{\varepsilon_R^*}{3.7\,\lambda_{(0)}} + \frac{2.51}{\lambda_{(0)}^{3/2} R_R \sqrt{f_R}} \right) \right]^{-2/5}, \quad \lambda_{(0)} = 1$$

$$\lambda_{(2)} = \left[-2\sqrt{f_R}\,\log\left(\frac{\varepsilon_R^*}{3.7\,\lambda_{(1)}} + \frac{2.51}{\lambda_{(1)}^{3/2} R_R \sqrt{f_R}} \right) \right]^{-2/5}$$

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$$\lambda_{(i)} = \left[-2\sqrt{f_R}\,\log\left(\frac{\varepsilon_R^*}{3.7\,\lambda_{(i-1)}} + \frac{2.51}{\lambda_{(i-1)}^{3/2} R_R \sqrt{f_R}} \right) \right]^{-2/5}, \quad i = 1,2\ldots,n$$

where n denotes the number of iterations.

Let's consider $R_R \geq 10^4$ which is approximately the smallest value of the Reynolds number in the rough turbulent domain and which corresponds to the largest value of the relative roughness $\varepsilon_R^* = 0.05$ (Moody, 1944). Let's varying also the relative roughness ε_R^* from 0.0001 to 0.05, which is the actual range in the rough turbulent zone as one can observe in the Moody chart (1944). It is found that for the third iteration according to the previous process, the deviation between $\lambda_{(1)}$ and $\lambda_{(3)}$ is less than $0.01\lambda_{(1)}$, i.e. $(\lambda_{(3)} - \lambda_{(1)}) / \lambda_{(1)} < 1\%$. This important result proves that the correction factor λ can be reasonably related to both Reynolds number R_R and ε_R^* through the following explicit relationship:

$$\lambda = \left[-2\sqrt{f_R}\,\log\left(\frac{\varepsilon_R^*}{3.7} + \frac{2.51}{R_R \sqrt{f_R}} \right) \right]^{-2/5} \tag{35}$$

EXAMPLE 1

Compute the diameter D of a partially filled circular conduit for the following data:

Discharge Q = 0.08 m^3/s; Absolute roughness ε = 0.00015 m; Channel bed slope S_0 = 0.0005; Aspect ratio $\eta = h/D$ = 0.6; Kinematic viscosity $\nu = 10^{-6}$ m^2/s

Solution

Let us first check if the flow is not in the rough turbulent domain. For this, let us use the following Hager's inequality (1987):

$$\varepsilon \geq 30\,\nu\left[Q(gS_0)^2\right]^{-0.2} \tag{36}$$

If this inequality is satisfied, then one can consider that the flow is rough turbulent. Thus:

$$30\times 10^{-6}\times\left[0.08\times(9.81\times 0.0005)^2\right]^{-0.2} = 0.00041711\ m > \varepsilon = 0.00015\ m$$

Hager's inequality (36) is not satisfied, which indicates that the flow is not in the rough turbulent domain. It is probably in the transitional zone.

The water area A can be written as:

$$A = \frac{D^2}{4}\left(\alpha - \sin\alpha\cos\alpha\right) \tag{37}$$

where:

$$\alpha = \cos^{-1}(1-2\eta) = \cos^{-1}(1-2\times 0.6) = 1.77215425\ rd \tag{38}$$

For circular conduit, table 1 gives:

$$A_1 = \left(\alpha - \sin\alpha\cos\alpha\right)/4 = \left[1.77215425 - \sin(1.77215425)\cos(1.77215425)\right]/4$$

Whence:

$$A_1 = 0.49202836$$

On the other hand, the wetted perimeter P can be written as:

$$P = \alpha D \tag{39}$$

Thus, according to table 1:

$$P_1 = \alpha = 1.77215425$$

According to Eq. (8), the function φ is as:

$$\varphi = \frac{A_1 Q}{P_1^2 \sqrt{g S_0}\, \varepsilon^{5/2}} = \frac{0.49202836 \times 0.08}{1.77215425^{\,2} \times \sqrt{9.81 \times 0{,}0005} \times 0.00015^{\,5/2}} = 649425633$$

Eq. (7) then becomes:

$$649425633 - 4\sqrt{2}\beta^{5/2} \log(14.8\beta) = 0 \tag{40}$$

The root of Eq. (40), obtained using an iterative process, is:

$$\beta = 948.161102$$

which is also the exact analytical solution obtained using the Lambert W-function. Application of Eq. (14) gives an approximate value of $\mathbf{W}_0(x) = 23.87002146$ leading to a value of $\beta = 947.078$. The deviation from the exact value is only 0.11 %.

According to Eq. (9), the characteristic length $L_R = D_R$ when the flow is assumed to be in the rough turbulent regime is:

$$L_R = D_R = \frac{\beta\, \varepsilon\, P_1}{A_1} = \frac{948.161102 \times 0.00015 \times 1.77215425}{0.49202836} = 0.51225332 \ m$$

Consequently, the hydraulic diameter $D_{h,R}$ is:

$$D_{h,R} = 4 D_R \frac{A_1}{P_1} \tag{41}$$

Whence:

$$D_{h,R} = \frac{4 \times 0.51225332 \times 0.49202836}{1.77215425} = 0.56889666 \ m$$

According to Eq. (4), the relative roughness is:

$$\varepsilon_R^* = \frac{\varepsilon}{D_{h,R}} = \frac{0.00015}{0.56889666} = 0.00026367$$

Hence, Eq. (3) gives:

$$f_R^{-1/2} = -2\log\left(\frac{\varepsilon_R^*}{3.7}\right) = -2 \times \log\left(\frac{0.00026367}{3.7}\right) = 8.2942877$$

That is:

$$f_R = 0.0145359$$

The Reynolds number R_R is given by Eq. (6) as:

$$R_R = \frac{4Q}{L_R P_1 \nu} = \frac{4Q}{D_R P_1 \nu} = \frac{4 \times 0.08}{0.51225332 \times 1.77215425 \times 10^{-6}} = 352503.698$$

Therefore, according to Eq. (35), the correction factor λ is as:

$$\lambda = \left[-2\sqrt{f_R} \log\left(\frac{\varepsilon_R^*}{3.7} + \frac{2.51}{R_R \sqrt{f_R}} \right) \right]^{-2/5}$$

$$= \left[-2 \times \sqrt{0.0145359} \times \log\left(\frac{0.00026367}{3.7} + \frac{2.51}{352503.698 \times \sqrt{0.0145359}} \right) \right]^{-2/5}$$

That is:

$$\lambda = 1.0264644$$

Since λ is greater than 1, one can conclude that the flow is in the transition zone.

Finally, according to Eq. (15), the required characteristic length $L_T = D_T = D$ is:

$$L_T = D = \lambda L_R = \lambda D_R = 1.0264644 \times 0.51225332 = 0.52580979\ m \approx 0.526\ m$$

Note that by an iterative process operated on Eq. (34), it was determined the exact value of λ was being:

$$\lambda_e = 1.02507433$$

The subscript "*e*" denotes "*exact*".

Thus, the deviation between the exact value of λ and the approximate value is:

$$100 \times \left| \frac{1.02507433 - 1.0264644}{1.0264644} \right| = 0.135\ \%$$

It is the same deviation observed when computing the diameter D. This can be written as:

$$D = \lambda_e D_R = 1.02507433 \times 0.51225332 = 0.52509773\ m \approx 0.525\ m$$

Checking calculations

Let's compute the diameter D by applying the classical method based on the Darcy-Weisbach formula. It is easy to show that the diameter D of a partially filled circular conduit can be written as:

$$D = \left[\frac{8f}{g} \frac{\alpha}{(\alpha - \sin\alpha \cos\alpha)^3} \frac{Q^2}{S_0} \right]^{1/5} \tag{42}$$

The friction factor *f* is given by Eq. (30) as:

$$f = \lambda_e^5 \, f_R \tag{43}$$

Whence:

$$f = 1.02507433^5 \times 0.0145359 = 0.016452$$

Finally, Eq. (42) gives:

$$D = \left[\frac{8 \times 0.016452}{9.81} \times \frac{1.77215425}{\left(1.77215425 - \sin(1.77215425) \times \cos(1.77215425)\right)^3} \times \frac{0.08^2}{0.0005} \right]^{1/5}$$

That is:

$$D = 0.52509775 \; m \approx 0.525 \; m$$

Thus, the deviation between the exact value of *D* and the approximate one is:

$$100 \times \left| \frac{0.52509775 - 0.52580979}{0.52580979} \right| = 0.135\,\%$$

One can conclude that the previous calculations using the advocated approach are verified.

EXAMPLE 2

Compute the depth *h* in a triangular channel for the following data:

$Q = 0.2\ m^3/s$; $\varepsilon = 0.0001\ m$; $S_0 = 0.0005$; $\alpha = 45°$ ($m = 1$); $\nu = 0.000001\ m^2/s$

Solution

Hager's inequality (36) is not verified since:

$$30 \times 10^{-6} \times \left[0.2 \times (9.81 \times 0.0005)^2\right]^{-0.2} = 0.00034726 \; m > \varepsilon = 0.0001 \, m$$

Thus, the flow is not in the rough turbulent domain. It is in the transition zone.

The water area can be written as:

$$A = mh^2 \tag{44}$$

As the chosen characteristic length is the flow depth *h*, then one can write from table 1 that:

$A_1 = m = 1$

The wetted perimeter is as:

$$P = 2h\sqrt{1+m^2} \tag{45}$$

Whence:

$$P_1 = 2\sqrt{1+m^2} = 2\sqrt{2}$$

According to Eq. (8) the function φ is:

$$\varphi = \frac{A_1 Q}{P_1^2\sqrt{gS_0}\,\varepsilon^{5/2}} = \frac{1\times 0.2}{8\times\sqrt{9.81\times 0.0005}\times 0.0001^{5/2}} = 3569607807$$

Thus, the implicit Eq. (7) becomes:

$$3569607807 - 4\sqrt{2}\beta^{5/2}\log(14.8\beta) = 0 \tag{46}$$

Adopting an iterative process, Eq. (46) gives β as:

$$\beta = 1825.54226$$

The approximate value of the Lambert function computed from Eq. (14) is $\mathbf{W}_0(x) = 25.5079388$ for which $\beta = 1823.5644$. The deviation from the exact value is of 0.11 % only.

According to Eq. (9), one may write:

$$L_R = h_R = \frac{\beta\varepsilon P_1}{A_1} = \frac{1825.54226\times 0.0001\times 2\times\sqrt{2}}{1} = 0.51634133\ m$$

This is the flow depth when the flow is assumed to be in the rough turbulent zone.

The hydraulic diameter $D_{h,R}$ is:

$$D_{h,R} = 4h_R\frac{A_1}{P_1} = 4\times 0.51634133\times\frac{1}{2\times\sqrt{2}} = 0.73021691$$

According to Eq. (4), the relative roughness is:

$$\varepsilon_R^* = \frac{\varepsilon}{D_{h,R}} = \frac{0.0001}{0.73021691} = 0.00013695$$

Hence, Eq. (3) gives:

$$f_R = \left[-2\log\left(\frac{\varepsilon_R^*}{3.7}\right)\right]^{-2} = \left[-2\times\log\left(\frac{0.00013695}{3.7}\right)\right]^{-2} = 0.01272941$$

The Reynolds number R_R is given by Eq. (6) as:

$$R_R = \frac{4Q}{L_R P_1 \nu} = \frac{4Q}{h_R P_1 \nu} = \frac{4\times 0.2}{0.51634133\times 2\times\sqrt{2}\times 10^{-6}} = 547782.442$$

Therefore, according to Eq. (35), the correction factor λ is as:

$$\lambda = \left[-2\sqrt{f_R}\,\log\left(\frac{\varepsilon_R^*}{3.7} + \frac{2.51}{R_R\sqrt{f_R}}\right)\right]^{-2/5}$$

$$= \left[-2\times\sqrt{0.01272941}\times\log\left(\frac{0.00013695}{3.7} + \frac{2.51}{547782.442\times\sqrt{0.01272941}}\right)\right]^{-2/5}$$

That is:

$$\lambda = 1.03059886$$

Since λ is greater than 1, one can conclude that the flow is in the transition zone.

According to Eq. (15), the required characteristic length $L_T = h_T = h$ is:

$$L_T = h = \lambda L_R = \lambda h_R = 1.03059886\times 0.51634133 = 0.53214078\ m \approx 0.532\ m$$

Performing an iterative process on Eq. (34), results in the following exact value of λ:

$$\lambda_e = 1.02903144$$

Thus, the exact value of the depth is:

$$h_e = \lambda_e h_R = 1.02903144\times 0.51634133 = 0.53133146\ m \approx 0.531\ m$$

Therefore, the deviation between the exact value of h and the approximate one is:

$$100\times\left|\frac{0.53133146 - 0.53214078}{0.53214078}\right| = 0.152\ \%$$

It can therefore be concluded that the calculations made with the proposed method are justifiable.

Checking calculations

The calculations can be verified by applying the Darcy-Weisbach formula to the considered triangular channel. Thus, one may easily derive the following relationship:

$$h=\left(\frac{f}{4g}\frac{\sqrt{1+m^2}}{m^3}\frac{Q^2}{S_0}\right)^{1/5} \tag{47}$$

According to Eq. (30), the friction factor f is such that:

$$f=\lambda_e^5\ f_R=1.02903144^{\,5}\times 0.01272941=0.01468763$$

Thus, Eq. (47) becomes:

$$h=\left(\frac{0.01468763}{4\times 9.81}\times\frac{\sqrt{2}}{1}\times\frac{0.2^2}{0.0005}\right)^{1/5}=0.53133144\ m\approx 0.531\,m$$

This result corroborates the excellent approximation of the calculations carried out.

CHARACTERISTIC LENGTH IN THE SMOOTH REGIME

Let's L_S be the characteristic length in the smooth flow regime corresponding to $\varepsilon^*\to 0$. One may write between the characteristic lengths L_S and L_R the following relationship which is homologous to Eq. (15):

$$L_S=\psi\,L_R \tag{48}$$

The coefficient ψ can be interpreted as the correction factor of the linear dimension when the flow is in the smooth regime.

Consider L_R as the characteristic length in the rough turbulent regime when the relative roughness is arbitrarily chosen such that $\varepsilon_R^*=0.01$. Thus, the corresponding friction factor f_R is given by Eq. (3) as:

$$f_R=0.03790371 \tag{49}$$

In the case of a rough turbulent flow regime, when combining both Eq. (3) and Darcy-Weisbach relationship, results in the following conveyance equation:

$$\frac{Q}{\sqrt{S_0}}=4\sqrt{2g}\,L_R^{5/2}\,\frac{A_1^{3/2}}{P_1^{1/2}}\log\left(14.8\,\frac{L_R\,A_1}{\varepsilon\,P_1}\right) \tag{50}$$

Eq. (50) is valid for any channel shape provided the flow is in the rough turbulent domain. According to Eqs. (4) and (5), and bearing in mind that $\varepsilon_R^* = 0.01$, one may write:

$$\frac{L_R A_1}{\varepsilon P_1} = \frac{1}{0.04} \tag{51}$$

Inserting this result into Eq. (50) and simplifying results in:

$$\frac{Q}{\sqrt{S_0}} = 14.528 \sqrt{g}\, L_R^{5/2} \frac{A_1^{3/2}}{P_1^{1/2}} \tag{52}$$

Therefore, the characteristic length L_R is expressed as:

$$L_R = 0.343 \left(\frac{Q}{\sqrt{g\, S_0}} \right)^{2/5} \left(\frac{P_1}{A_1^3} \right)^{1/5} \tag{53}$$

Furthermore, according to both Eqs. (48) and (53), the characteristic length L_S for any shape of channel carrying flow in a smooth state is:

$$L_S = 0.343 \psi \left(\frac{Q}{\sqrt{g\, S_0}} \right)^{2/5} \left(\frac{P_1}{A_1^3} \right)^{1/5} \tag{54}$$

REYNOLDS NUMBER IN THE SMOOTH REGIME

The Reynolds number R_S characterizing the flow in the smooth regime can be expressed by relations similar to Eqs. (17), (18) and (19). Thus, one may write:

$$R_S = \psi^{-1} R_R \tag{55}$$

The Reynolds number R_R can be expressed when combining Eqs. (6) and (53). After some rearrangements, the final result is:

$$R_R = \frac{11.67}{\nu} \left(g\, S_0\, Q^3 \right)^{1/5} \left(\frac{\sqrt{A_1}}{P_1} \right)^{6/5} \tag{56}$$

DIMENSIONLESS CORRECTION FACTOR ψ AND RESULTING L_S RELATIONSHIP

To work out the relationship expressing ψ, the same theoretical approach used to determine λ remains valid resulting in what follows:

$$\psi = \left(\frac{f_S}{f_R}\right)^{1/5} \tag{57}$$

Taking into account the result given by Eq. (49), Eq. (57) is reduced to:

$$f_S^{-1/2} = 5.13640345\ \psi^{-5/2} \tag{58}$$

On the other hand, $f_S^{-1/2}$ is expressed by Eq. (1) for $\varepsilon^* \to 0$ such that:

$$f_S^{-1/2} = -2\log\left(\frac{2.51}{R_S\sqrt{f_S}}\right) \tag{59}$$

Combining Eqs. (55), (58) and (59) yields:

$$\psi^{5/2}\log\left(\frac{\psi^{3/2}\ R_R}{12.8924}\right) = 2.5682 \tag{60}$$

The known parameter of Eq. (60) is R_R in accordance with Eq. (56). What is sought is the parameter ψ. The implicit nature of Eq. (60) requires a trial-and-error method. However, it has been found that the following relationship is an excellent approximate form of Eq. (60):

$$\psi = \frac{1.918}{\sqrt{\log\left(R_R\right)}} \tag{61}$$

The maximum deviation observed between Eqs. (60) and (61) is less than 0.4% for $R_R >$ 4770, which confirms perfectly the appropriateness of the fit of the derived Eq. (61). The relative error close to 0.4% is obtained for large values of R_R such as $R_R \geq 10^7$. According to the fundamental Eq. (48), it is the same relative error that will be committed on the computation of the characteristic length L_S since no error affects the calculation of L_R.

It is worth noting that Eq. (60) gives $\psi = 1$ for $R_R \approx 4770$, meaning that $f_S = f_R$ in accordance with Eq. (57). It is therefore recommended to apply Eq. (60) for values of R_R such as $R_R > 4770$. The calculations performed on the implicit Eq. (60) have shown that the values of ψ are less than unity for $R_R > 4770$ which means that $L_S < L_R$ according to Eq. (48).

Finally inserting Eq. (61) into Eq. (54), the characteristic length L_S of any shape of a channel carrying flow in a smooth state is expressed by the following explicit relationship:

$$L_S = \frac{0.6576}{\sqrt{\log(R_R)}}\left(\frac{Q}{\sqrt{g\,S_0}}\right)^{2/5}\left(\frac{P_1}{A_1^3}\right)^{1/5} \tag{62}$$

EXAMPLE 3

Compute the diameter D of a partially filled circular conduit, characterized by an absolute roughness $\varepsilon \to 0$, for the following data:

Discharge $Q = 0.2\ m^3/s$; Conduit slope $S_0 = 0.0004$; Aspect ratio $\eta = h/D = 0.6$; Kinematic viscosity $\nu = 10^{-6}\ m^2/s$.

Solution

On one hand, one may write that:

$$\alpha = \cos^{-1}(1-2\eta) = \cos^{-1}(1-2\times 0.6) = 1.77215425\ rd$$

For circular conduit, table 1 gives:

$$A_1 = (\alpha - \sin\alpha\cos\alpha)/4 = [1.77215425 - \sin(1.77215425)\cos(1.77215425)]/4$$

whence:

$$A_1 = 0.49202836$$

On the other hand, according to Eq. (39) the wetted perimeter P is expressed as:

$$P = \alpha\, D \tag{39}$$

Thus, with regard to table 1:

$$P_1 = \alpha = 1.77215425$$

According to Eq. (56), the Reynolds number R_R is as:

$$R_R = \frac{11.67}{\nu}\left(g\,S_0\,Q^3\right)^{1/5}\left(\frac{\sqrt{A_1}}{P_1}\right)^{6/5}$$

$$= \frac{11.67}{10^{-6}}\left(9.81\times 0.0004\times 0.2^3\right)^{1/5}\times\left(\frac{\sqrt{0.49202836}}{1.77215425}\right)^{6/5} = 482421.782$$

Furthermore, the required value of $L_S = D$ is given explicitly by Eq. (62) as:

$$L_S = \frac{0.6576}{\sqrt{\log(R_R)}}\left(\frac{Q}{\sqrt{g\,S_0}}\right)^{2/5}\left(\frac{P_1}{A_1^3}\right)^{1/5}$$

$$= \frac{0.6576}{\sqrt{\log(482421.782)}}\times\left(\frac{0.2}{\sqrt{9.81\times 0.0004}}\right)^{2/5}\times\left(\frac{1.77215425}{0.49202836^3}\right)^{1/5} = 0.75305915\ m \approx 0.753\,m$$

Checking calculations

For the value of R_R computed above, an iterative procedure operated on the implicit Eq. (60) gives the exact value of the dimensionless correction factor ψ as:

$$\psi_e = 0.80399259$$

On the other hand, according to Eq. (53), the characteristic length L_R is:

$$L_R = 0.343\left(\frac{Q}{\sqrt{g\,S_0}}\right)^{2/5}\left(\frac{P_1}{A_1^3}\right)^{1/5}$$

$$= 0.343\times\left(\frac{0.2}{\sqrt{9.81\times 0.0004}}\right)^{2/5}\times\left(\frac{1.77215425}{0.49202836^3}\right)^{1/5} = 0.93641108\ m \approx 0.936\,m$$

Thus, Eq. (48) gives the exact value of L_S as:

$$L_{S,e} = \psi_e\,L_R = 0.80399259\times 0.93641108 = 0.75286757\ m \approx 0.753\,m$$

The approximate and exact values of L_S are extremely close, which confirms the reliability of the proposed method.

Another way to check calculations is to apply Eq. (42). Thus:

$$D = \left[\frac{8f}{g}\,\frac{\alpha}{(\alpha - \sin\alpha\cos\alpha)^3}\,\frac{Q^2}{S_0}\right]^{1/5} \qquad (42)$$

The friction factor f corresponds to f_S and can be deduced from Eq. (57) as:

$$f = f_S = \psi_e^5\,f_R$$

That is:

$$f_S = 0.80399259^5\times 0.03790371 = 0.01273333$$

Finally, Eq. (37) gives:

$$D = \left[\frac{8 \times 0.0127333}{9.81} \times \frac{1.77215425}{\left(1.77215425 - \sin(1.77215425) \times \cos(1.77215425)\right)^3} \times \frac{0.2^2}{0.0004} \right]^{1/5}$$

Whence:

$$D = 0.75256292\ m \approx 0.753\ m$$

The deviation between this value and the approximate one, i.e. 0.75305915, is 0.0659% only.

CONCLUSION

The study focused in-depth on two essential parameters of the flow in the transition and smooth domains. This is how the characteristic length, as well as the Reynolds number, were drawn up and expressed under practical relationships. The characteristic length can be either the width of a rectangular channel, the diameter of a circular pipe, or the depth of flow in any channel. Currently, this characteristic length is determined by an iterative process such as the laborious trial and error method. To determine the characteristic length in the studied flow domains, it was initially assumed that the flow regime is rough turbulent associated with a characteristic length denoted L_R. This is governed by an implicit relationship whose analytical resolution can be eased by Lambert's functions as it has been shown through some numerical examples. To compute the characteristic length L_T sought in the transition domain, the length L_R is corrected for effects of a dimensionless correction coefficient λ [Eq. (15)] which depends on the Reynolds number and on the relative roughness in the rough domain [Eq. (35)]. Similarly, the characteristic length L_S of the smooth flow is obtained by correcting the characteristic length L_R for effects of a dimensionless correction coefficient denoted ψ [Eq. (48)]. The ψ coefficient has been shown to be closely related to the known Reynolds number R_R of rough turbulent flow [Eq. (61)].

The Reynolds numbers in the transition and smooth domains have been expressed by practical relationships according to known parameters. The Reynolds number R_T characterizing the transition flow is expressed by Eq. (19) along with Eq. (35), while the Reynolds number R_S in the smooth regime is governed by Eq. (55).

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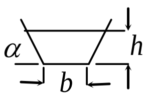
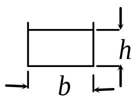
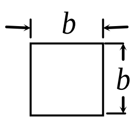
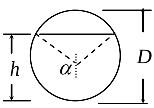
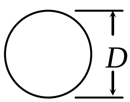
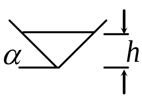
APPENDIX

Table 1: Formulae governing the dimensionless parameters A_1 and P_1 for some channel shapes

Channel shape	Aspect ratio	A	P	L	A_1	P_1
	$\eta = b/h$ $m=\text{cotg}\,\alpha$	$bh+mh^2$	$b+2h(1+m^2)^{1/2}$	b	$\eta^{-1}(1+m\eta^{-1})$	$1+2\eta^{-1}(1+m^2)^{1/2}$
				h	$\eta + m$	$\eta+2(1+m^2)^{1/2}$
	$\eta = b/h$ $m = 0$	bh	$b + 2h$	b	η^{-1}	$1 + 2\eta^{-1}$
				h	η	$\eta + 2$
	$\eta = 1$	B^2	$4B$	B	1	4
	$\eta = h/D$ $\alpha=\cos^{-1}(1-2\xi)$	$D^2(\alpha\text{-SIN}\alpha\ \text{COS}\alpha)/4$	αD	D	$(\alpha\text{-}\sin\alpha\ \cos\alpha)/4$	α
				H	$(\alpha\text{-SIN}\alpha\ \text{COS}\alpha)/(4\eta^2)$	α/η
	$\eta = 1$	$\pi D^2/4$	πD	D	$\pi/4$	π
	$\eta = 0$ $m=\text{cotg}\,\alpha$	mh^2	$2h(1+m^2)^{1/2}$	h	m	$2(1+m^2)^{1/2}$