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NEW THEORETICAL CONSIDERATIONS ON THE GRADUALLY VARIED FLOW IN A TRIANGULAR CHANNEL

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ABSTRACT

Proper sizing of a channel, whatever its shape depends on the knowledge and mastering of the varied flow that occurs in it. This is, in particular, the case of the triangular channel on which the present study is focused. In the field of gradually varying flow, previous studies have mainly focused on rectangular channels. Very few studies have been carried out on other profiles, in particular the triangular profile. The differential equation that governs the gradually varied flow is transformed into a dimensionless generalized equation using a rigorous theoretical approach. Carefully applied to the triangular channel, it was possible to plot dimensionless backwater and drawdown curves for the special cases corresponding to zero and critical slopes

For the case of the C1-type backwater curve, a fast approximate method is proposed allowing the simple calculation of the length that separates two given depths.

Keywords: Gradually varied flow, GVF, triangular channel, critical slope, horizontal channel, backwater curve, drawdown curve.

INTRODUCTION

The fluid flow is a uniform flow if the flow parameters remain constant with distance along the flow path. The fluid flow is nonuniform if the flow parameters vary and are different at different points on the flow path (Lencastre, 1999; Henderson, 1966; Chow, 1959). If the nonuniformity is low, the flow will be described as gradually varying. A permanent flow along a channel can be a succession of uniform flows that gradually and rapidly vary if the channel bottom has discontinuities or changes. The gradually varied flow remains a permanent flow, i.e., the flow rate remains constant over time. On the other hand, changes in flow sections, generally caused by changes in slope, make the flow nonuniform. The transitions will be considered to take place over a relatively long distance, which explains the term gradual. The study of gradually varied flow is based on the use of a fundamental theorem of fluid mechanics in this case "Bernoulli's theorem" which is based on the principle of energy conservation (Lencastre, 1999; Henderson, 1966).

The term "Backwater Curve" is used herein as the longitudinal profile of the water surface in a nonuniform steady flow in an open prismatic channel (Chen, and Wang, 1969; Valentine, 1967). In fact, when the depth of water increases in the direction of flow, i.e., the variation in the depth according to the distance dh/dL is positive; then, the surface profile is classified as a backwater curve, while when the depth decreases, i.e., the variation in the depth according to the distance dh/dL is negative; then, the surface profile is known as the drawdown curve (Valentine, 1964; 1967), US Geological Survey (1955), and US Army Corps of Engineers (1959). Backwater curves, or the water surface profiles, can be classified according to the slope S_0 of the canal. There is, for a given flow rate, a channel slope S_c for which flow occurs at the critical depth. The corresponding backwater curves form the C-type group. Slopes of the canal less than this critical slope will be considered low. It is said to be a mild slope. This will form the M-type backwater curve group. The channel slopes greater than the critical slope are so-called steep slopes and form the S-type backwater curve group. The horizontal slopes are associated with the type H group, while the adverse slopes correspond to the A-type group.

The computation of the water surface profiles is based on the principle of energy, that is, quite simply on the Bernoulli equation applied between two chosen flow sections. It is easily demonstrated that the final result is a differential equation called the gradually varied flow equation expressed as (Chen and Wang, 1969; Chow, 1955):

$$\frac{dh}{dL} = \frac{S_0 - S_f}{1 - \dfrac{Q^2 e}{gA^3}} \tag{1}$$

where Q is the flow rate, e is the top width, A is the water area, g is the acceleration due to gravity, h is the water depth at a given section, and dh/dL gives the variation of water depth along the channel in the flow direction. Note that dL, which is the distance between two given sections, is taken on the horizontal reference datum and not along the bottom channel.

Eq. (1) allows plotting of the free surface profile curves using different methods. Among the most widespread methods, one may distinguish the following three methods: method by successive approximations, method by direct integration, and method by graphical integration (Chow; 1959).

By integrating between two sections A_1 and A_2 of respective depths h_1 and h_2, equation (1) becomes:

$$L_2 - L_1 = \int_{h_1}^{h_2} f(h)dh \tag{2}$$

This last equation is difficult to solve analytically because the second member is a complex function. It is quite easy to solve for a few simple cases such as the case of the large width rectangular channel using the recent method of Achour and Amara (2021). However, the old methods are also effective but long such the preferred calculation method of Bresse that is well explained in Lencastre (1999), Bakhmeteff (1932) and Chow (1955; 1959).

The graphical integration method puts the differential equation of the gradually varied flow into the following form, provided the flow rate and the channel profile are known:

$$dL = \frac{dh}{f(h)} \tag{3}$$

That is:

$$L_2 - L_1 = \int_{L_1}^{L_2} dL = \int_{L_1}^{L_2} \frac{dh}{f(h)} \tag{4}$$

One thus obtains a first-order differential equation, integrable by means of a constant of integration known thanks to the boundary conditions.

The calculation methods usually used for gradually varied flow, such as those previously indicated, do not take into account the effect of the viscosity of the flowing liquid. As a result, their application would be reserved exclusively for gradually varied flows in a rough turbulent regime.

The present study proposes a new theoretical approach allowing the calculation as well as the plotting of the free surface of flow in a triangular channel. Few studies are reported in the literature specifically concerning this profile, but it is useful to recall the study of Vatankhah (2010). An analytical solution of the gradually varied-flow equation for triangular channels was derived based on Manning's formula. The proposed solution can accurately determine the flow profiles of triangular channels.

In this study, the differential equation governing the gradually varying flow is transformed into a function defined by dimensionless terms. The integration of this function leads to a mathematical formulation allowing the direct solution of all the problems of the gradually varying flow in a rough turbulent flow regime. Particular attention is given to the special cases of critical and horizontal slopes which are of remarkable mathematical interest. Backwater and drawdown curves are plotted in dimensionless terms, valid for any apex angle of the triangular channel.

RESULTING GRADUALLY VARIED FLOW EQUATION

The Manning-Strickler equation gives the mean velocity of a uniform flow in the following form (Strickler, 1923):

$$V = \frac{Q}{A} = k\, R_h^{2/3} S_f^{1/2} \tag{5}$$

where Q is the flow rate, A is the water area, k is the Strickler coefficient, and R_h is the hydraulic radius. Recall that S_0 is the slope of the channel and that S_f is the slope of the hydraulic grade line or the linear hydraulic head loss. For a flow of depth h flowing in a triangular channel of apex angle θ, one can write:

$$A = mh^2 \tag{6}$$

where $m = tg\ (\theta/2)$.

The wetted perimeter:

$$P = 2h\sqrt{1+m^2} \tag{7}$$

The hydraulic radius:

$$R_h = \frac{mh}{2\sqrt{1+m^2}} \tag{8}$$

Taking into account the above parameters, Eq. (5) gives:

$$S_f = \frac{2^{4/3}(1+m^2)^{2/3}}{k^2 m^{10/3} h^{16/3}} Q^2 \tag{9}$$

On the other hand, one may write:

$$\frac{Q^2 e}{gA^3} = \frac{2Q^2}{gm^2 h^5} \tag{10}$$

Inserting both Eqs. (9) and (10) into Eq. (1) results in:

$$dL = \frac{1 - 2Q^2 / (gm^2 h^5)}{S_0 - \dfrac{2^{4/3}(1+m^2)^{2/3}}{k^2 m^{10/3} h^{16/3}} Q^2} dh \tag{11}$$

When the flow is critical with depth h_c, Eq. (10) is then equal to unity. One may deduce that:

$$Q^2 = \frac{1}{2} gm^2 h_c^5 \tag{12}$$

Eliminating Q^2 between Eqs. (11) and (12) and rearranging results in:

$$dL^* = \frac{1 - h^{*-5}}{S_0 - \dfrac{\alpha g}{k^2 h^{*16/3} h_c^{1/3}}} dh^* \tag{13}$$

where:

$$L^* = L / h_c \tag{14}$$

$$h^* = h / h_c \tag{15}$$

$$\alpha = \frac{2^{1/3}(1+m^2)^{2/3}}{m^{4/3}} \tag{16}$$

Multiplying the numerator and the denominator of Eq. (13) by the quantity ($k^2 h^{*16/3} h_c^{1/3}$) yields:

$$dL^* = \frac{\left(1 - h^{*-5}\right) k^2 h^{*16/3} h_c^{1/3}}{S_0 k^2 h^{*16/3} h_c^{1/3} - \alpha g} dh^* \tag{17}$$

Using Strickler's Eq. (5), one can easily derive the critical bottom slope S_c for the triangular channel as:

$$S_c = \frac{\alpha g}{k^2 h_c^{1/3}} \tag{18}$$

Let δ be $\delta = 1/S_c$, which amounts to writing that:

$$\delta = \frac{k^2 h_c^{1/3}}{\alpha g} \tag{19}$$

By virtue of Eq. (19) and some arrangements, Eq. (17) becomes:

$$dL^* = \delta \frac{h^{*16/3} - h^{*1/3}}{S_0 \delta h^{*16/3} - 1} dh^* \tag{20}$$

Introducing the following dimensionless number:

$$\beta = S_0 \delta = S_0 / S_c \tag{21}$$

Eq. (21) is then rewritten as:

$$dL^* = \delta \frac{h^{*16/3} - h^{*1/3}}{\beta h^{*16/3} - 1} dh^* \tag{22}$$

Integrating Eq. (22) between two relative depths h_1^* and h_2^* yields:

$$\overset{h_2^*}{\underset{h_1^*}{L^*}} = \delta \int_{h_1^*}^{h_2^*} \frac{h^{*16/3} - h^{*1/3}}{\beta h^{*16/3} - 1} dh^* \tag{23}$$

Consider the following function:

$$F(h^*, \beta) = \int_{h_1^*}^{h_2^*} \frac{h^{*16/3} - h^{*1/3}}{\beta h^{*16/3} - 1} dh^* \tag{24}$$

One may then write what follows:

$$\overset{h_2^*}{\underset{h_1^*}{L^*}} = \delta \left[F(h_2^*, \beta) - F(h_1^*, \beta) \right] \tag{25}$$

That is to say:

$$\overset{h_2}{\underset{h_1}{L}} = h_c \delta \left[F(h_2^*, \beta) - F(h_1^*, \beta) \right] \tag{26}$$

Thus, it is the function $F(h^*, \beta)$ expressed by Eq. (24) which makes it possible to solve the problems of the gradually varied flow in a triangular channel.

THE DIMENSIONLESS PARAMETER β

By definition and for a uniform flow for which $S_0 = S_f$, one may write:

$$h = h_n \tag{27}$$

where h_n is the normal depth.

Dividing the two members of Eq. (27) by h_c results in:

$$h^* = h_n^* \tag{28}$$

In uniform flow regime, Eq. (7) is written as:

$$S_0 = \zeta \frac{Q^2}{k^2\, h_n^{16/3}} \tag{29}$$

where ζ depends solely on m as:

$$\zeta = \frac{2^{4/3}(1+m^2)^{2/3}}{m^{10/3}} \tag{30}$$

Inserting Eq. (12) into Eq. (29) and rearranging results in:

$$S_0 = \frac{\alpha\, g}{k^2\, h_n^{*\,16/3} h_c^{\,1/3}} \tag{31}$$

where α is defined by Eq. (15).

Combining Eqs. (19), (21), and (31) yields:

$$\beta = S_0\, \delta = h_n^{*\,-16/3} \tag{32}$$

It is thus shown that the dimensionless parameter β is closely related to the normal relative depth. This implies that β is independent of the depth h and that it is constant for a given case. Eq. (32) has never been established before. A similar equation was derived by Achour and Amara (2021) for the case of a wide rectangular channel.

The interest and advantage of Eq. (32) lies in the fact that it allows the classification of slopes. This is done as follows:

i) $\beta > 1$, $h_n^* < 1$, i.e., $h_n < h_c$

This means that the normal flow regime is supercritical and the slope is of steep type (S).

ii) $\beta = 1$, $h_n^* = 1$, i.e. $h_n = h_c$

This means that the normal flow regime is critical, which implies that the slope is of the critical type (C).

iii) $0 < \beta < 1$, $1 < h_n^* < \infty$, $h_c < h_n < \infty$

This means that the normal flow regime is subcritical, which implies that the slope is of the Mild type (M).

iv) $\beta = 0$, $h_n^* = \infty$, $\delta \neq 0$, so $S_0 = 0$

The normal flow regime does not exist and the slope is horizontal.

v) $\beta < 0$, $h_n^* < 0$, so $h_n < 0$ and $S_0 < 0$

The normal flow regime does not exist and the slope is adverse (A).

SPECIAL CASES

Horizontal slope

This case corresponds to $S_0 = 0$ and Eq. (20) becomes:

$$dL^* = \delta\left(h^{*1/3} - h^{*16/3}\right)dh^* \quad (33)$$

Integrating this equation between two relative depths h_1^* and h_2^* results in:

$$\int_{h_1^*}^{h_2^*} dL^* = \delta \int_{h_1^*}^{h_2^*} \left(h^{*1/3} - h^{*16/3}\right)dh^* \quad (34)$$

which gives:

$$\left. L \right|_{h_1}^{h_2} = h_c\,\delta\left(\frac{3}{4}h^{*4/3} - \frac{3}{19}h^{*19/3} + C_0\right)\Bigg|_{h_1}^{h_2} \quad (35)$$

Eq. (35) can be rewritten as:

$$\underset{h_1}{\overset{h_2}{L}} = h_c \delta \left[F(h_2^*, 0) - F(h_1^*, 0) \right] \tag{36}$$

where:

$$F\left(h^*, 0\right) = \frac{3}{4} h^{*\,4/3} - \frac{3}{19} h^{*\,19/3} + C_0 \tag{37}$$

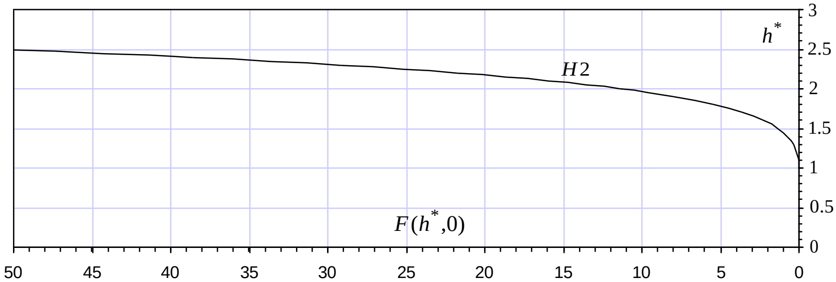
C_0 is an integration constant that can be determined by the initial conditions. According to Eq. (23), for $h^* = 1$ and $\beta = 0$ one may write F (1, 0) = 0. Inserting this result into Eq. (37) yields:

C_0 = (3/19) – (3/4) = - 45/76

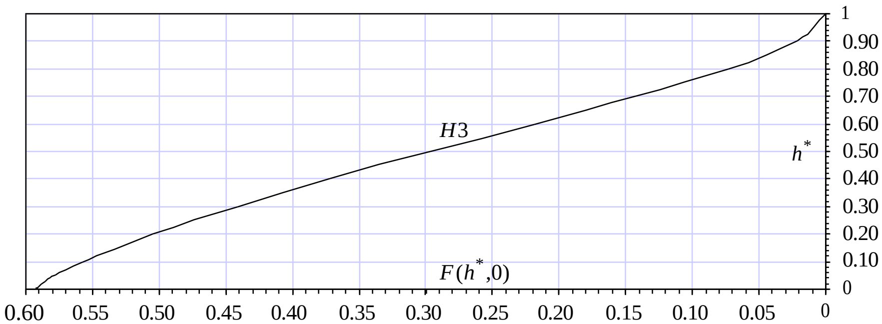
Therefore, Eq. (37) can be written as:

$$F\left(h^*, 0\right) = \frac{3}{4} h^{*\,4/3} - \frac{3}{19} h^{*\,19/3} - \frac{45}{76} \tag{38}$$

Eq. (38) is used to plot the H2-type backwater curve (*dh /dL* > 0) for h^* values greater than 1 ($h^* > 1$), as well as the H3-type drawdown curve (*dh /dL* < 0) for h^* values less than 1 ($h^* < 1$). These curves are shown in Fig. 1. When calculating the function $F\left(h^*, 0\right)$ using Eq. (38), the obtained values can be negative; one must then consider the absolute value.



a) $h^* > 1$



b) $h^* < 1$

Figure 1: Graphical representation of H2 and H3-type backwater and drawdown dimensionless curves respectively according to Eq. (38)

Eq. (36) along with Eq. (38) allows computing the length *L* separating two given depths h_1 and h_2 of a flow evolving in a horizontal triangular channel. Combining these two relationships results in:

$$\overset{h_2}{\underset{h_1}{L}} = h_c\,\delta\left[\frac{3}{4}\left(h_2^{*4/3} - h_1^{*\,4/3}\right) - \frac{3}{19}\left(h_2^{*19/3} - h_1^{*19/3}\right)\right] \tag{39}$$

Eq. (39) is the definitive relationship that expresses the length *L* separating two depths h_1 and h_2 of a flow evolving in a horizontal triangular channel.

Critical slope

In this case, on may write:

$$S_0 = S_c\,,\ \beta = S_0 / S_c = 1,\ h_n^* = 1, \text{ i.e. } h_n = h_c$$
$$\delta = 1 / S_c = 1 / S$$

Thus, Eq. (24) is reduced to:

$$F(h^*,1) = \int_{h_1^*}^{h_2^*} \frac{h^{*16/3} - h^{*1/3}}{h^{*16/3} - 1}\,dh^* \tag{40}$$

Thus, Eq. (26) is written as:

$$\overset{h_2}{\underset{h_1}{L}} = \frac{h_c}{S}\left[F(h_2^*,1) - F(h_1^*,1)\right] \tag{41}$$

Eq. (41) is the relationship that allows calculating the length L separating two given depths h_1 and h_2 of a flow evolving in a triangular channel with a critical slope.

Eq. (41) can be rewritten as:

$$L_{h_1}^{h_2} = \frac{h_c}{S} I \tag{42}$$

where:

$$I = \left[F(h_2^*, 1) - F(h_1^*, 1) \right] \tag{43}$$

I is closely related to the calculation of the integral expressed by Eq. (40).

The first case to study is for which:

$$h > h_n = h_c \text{ ; i.e. } h^* > 1$$

Eq. (22) becomes:

$$dL^* = \frac{1}{S} \frac{\left(h^{*16/3} - h^{*1/3} \right)}{h^{*16/3} - 1} dh^* \tag{44}$$

Since $h^*>1$, the numerator and the denominator of Eq. (44) are positive, i.e. $dL/dh > 0$.

If $h^* \to \infty$, then $dL/dh \to 1/S$, which allows us to conclude that the curve asymptotically approaches the horizontal.

For $h = h_c$, i.e. $h^*=1$, Eq. (44) leads to the following indeterminacy:

$$\frac{dL^*}{dh^*} = \frac{1}{S} \frac{0}{0}$$

Using L'Hopital's rule, the limit of dL/dh when h^* approaches 1 is:

$$\lim_{h^* \to 1} \frac{dL^*}{dh^*} = \frac{1}{S} \lim_{h^* \to 1} \frac{\frac{d}{dh^*}\left(h^{*16/3} - h^{*1/3} \right)}{\frac{d}{dh^*}\left(h^{*16/3} - 1 \right)} = \frac{1}{S} \left(\frac{16/3 - 1/3}{16/3} \right) = \frac{1}{S} \frac{15}{16} = \frac{0.9375}{S} \tag{45}$$

In this case, a C1-type backwater curve occurs (Fig. 2).

For a wide rectangular channel, Achour and Amara (2021) derived the following:

$$\lim_{h^* \to 1} \frac{dL^*}{dh^*} = \frac{0.9}{S} \tag{46}$$

The second case to study is for which:

$h < h_n = h_c$; i.e. $h^* < 1$

Since $h^* < 1$, the numerator and the denominator of Eq. (44) are both negative, *dL/dh* > 0. The water depth *h* increases downstream. For *h* = 0, i.e. $h^* = 0$, Eq. (44) becomes:

$$\frac{dL^*}{dh^*} = 0 \tag{47}$$

At this singular point of the flow, the depth becomes zero and the tangent of the curve is perpendicular to the bottom of the channel. This is the C3-type backwater curve shown in Fig. 2.

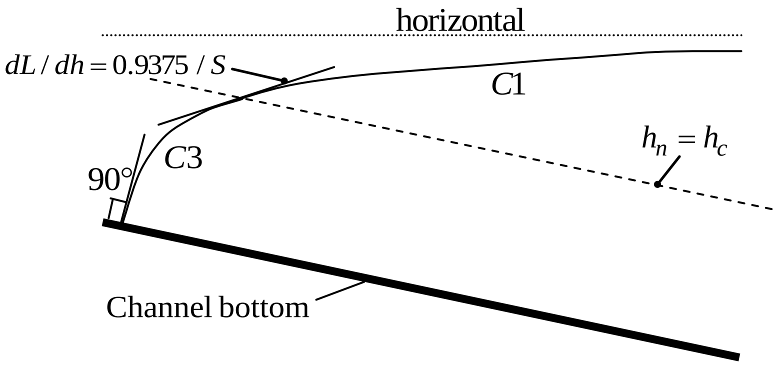


Figure 2: C-type backwater curves

The function $F(h^*, 1)$ corresponding to critical flow has been the subject of an in-depth regression study for the values $h^* < 1$ and $h^* > 1$.

For the values of h^* such as $h^* < 1$, the function $F(h^*, 1)$ expressed by Eq. (40) can be reasonably represented by the following relationship, obtained with a coefficient of determination $R^2 = 1$:

$$F\left(h^*, 1\right) = 0.7437\, h^{*\,1.3292} \tag{48}$$

The maximum deviation caused by Eq. (48) is only approximately 1.07 %. Eq. (48) allows a simplified and rapid calculation of the C3-type backwater curve, in particular the calculation of the length L separating two given depths h_1 and h_2, according to Eq. (26).

Taking into account Eq. (48) and knowing that $\beta = 1$, Eq. (26) is reduced to:

$$L_{h_1}^{h_2} = 0.7437\, h_c\, \delta\left(h_2^{*1.3292} - h_1^{*1.3292}\right) \tag{49}$$

Eq. (49) is the final form that allows us to quickly know the order of magnitude of the length L separating two given depths of a flow in a triangular channel characterized by a critical bed slope provided $h^* < 1$.

A more elaborate approximation for the definite integral (Eq. 40) can be obtained in the interval $0 < h^* < 1$, C-3 branch, using a Taylor series truncated to a suitable order. Integrating terms of the expansion series results in:

$$F\left(h^*,1\right) = \frac{3}{4}h^{*4/3} - \frac{3}{19}h^{*19/3} + \frac{3}{20}h^{*20/3} - \frac{3}{35}h^{*35/3} + \frac{1}{12}h^{*12} \tag{50}$$

The maximum deviation caused by the use of Eq. (50) is only 0.5 % reached for the extreme value $h^* = 0.99$.

On the other hand, for the values of h^* such as $h^* > 1$ and more precisely in the wide range $1 < h^* \le 2$, the function $F\,(h^*, 1)$ expressed by Eq. (40) can be governed with fairly great accuracy by the following relationship, obtained with a coefficient of determination $R^2 = 1$:

$$F\left(h^*,1\right) = 0.022\, h^{*2} + 0.9129\, h^* - 0.2013 \tag{51}$$

The maximum deviation caused by Eq. (51) is only 0.22 %, which allows for a fast and precise calculation of the C1-type backwater curve.

Moreover, combining both Eqs. (41) and (51) yields the following rearrangement:

$$L_{h_1}^{h_2} = (h_2^* - h_1^*)\left[0.022\,(h_2^* + h_1^*) + 0.9129\right]\frac{h_c}{S} \tag{52}$$

Relationship (52) causes a maximum deviation of 2% when the relative depth h approaches very close to the critical depth h_c, i.e., when the value of the relative depth h^* is between 1 and 1.01. For larger values of h_1^* and h_2^*, the deviation is very acceptable and sometimes insignificant.

The approximate relationship (52) is very simple and does not require any complicated calculations. It avoids the calculation of the integral expressed by the relationship (40). Eq. (52) can be used to quickly determine the order of magnitude of the length L separating two given depths h_1 and h_2 in a C1-type backwater curve evolving in a triangular channel, provided that the slope S of the channel and the critical depth are known.

If one desires to obtain more precision, a formal integration by use of Taylor's expansion series for Eq. (40) around $h^* = 1$ can be obtained. Applying the procedure as done before, one obtains:

$$F(h^*,1) = \frac{25}{32}h^* + \frac{5}{64}h^{*2} - \frac{115}{1728}(h^* - 1)^3 + \frac{175}{4608}(h^* - 1)^4 - \frac{3121}{311040}(h^* - 1)^5 \qquad (53)$$

When using Eq. (53), the resulting maximum deviation is insignificant since it is only 0.08 %, obtained for $h^* = 2$.

EXAMPLE 1

Let us consider a horizontal triangular channel of apex angle θ = 90°, corresponding to m = tg(45°) = 1, conveying a flow rate Q = 10 m^3/s. The absolute roughness characterizing the state of the inner wall of the channel is ε = 2 mm.

From the critical flow depth corresponding to a control section, calculate and draw the water surface profile over a length of L = 5 m following the control section.

Solution

First, it is necessary to suppose that the flow is in the rough turbulent domain which corresponds to the condition of application of the theoretical relationships derived previously.

Start from the control section, and then go up the liquid stream from downstream to upstream. The calculations will lead to the plot of the H2-type backwater curve.

According to Eq. (12), the critical flow depth is as follows:

$$h_c = \left(\frac{2Q^2}{g\,m^2}\right)^{1/5} = \left(\frac{2\times10^2}{9.81\times1^2}\right)^{1/5} = 1.82756233\,m$$

It is therefore considered that the first depth of the flow corresponds to:

$$h_1 = h_c = 1.82756233\,m$$

Thus:

$$h_1^* = h_1 / h_c = 1$$

For the purposes of the calculation, a depth step $\Delta h = 0.01m$ is considered. This means that the second depth to consider is:

$$h_2 = h_1 + \Delta h = 1.82756233 + 0.01 = 1.83756233\, m$$

Thus:

$$h_2^* = h_2 / h_c = 1.83756233 / 1.82756233 = 1.00547177$$

More generally, one may write:

$$h_{i+1} = h_i + \Delta h\, ,\, i = 1, 2, ...$$

The next step consists of calculating the dimensionless parameter δ according to Eq. (19), requiring the value of both the Strickler coefficient k and the parameter α.

The relationship that explicitly relates the Strickler roughness coefficient k to the absolute roughness ε in the rough turbulent flow regime is given by Achour and Amara (2022) as:

$$\frac{k\, \varepsilon^{1/6}}{8.315\sqrt{g}} = 1 \tag{54}$$

That is,

$$k = \frac{8.315\sqrt{g}}{\varepsilon^{1/6}} = \frac{8.315 \times \sqrt{9.81}}{0.002^{1/6}} = 73.3711103\, m^{1/3} / s$$

According to Eq. (16), the parameter α is as:

$$\alpha = \frac{2^{1/3}(1+m^2)^{2/3}}{m^{4/3}} = \frac{2^{1/3} \times (1+1^2)^{2/3}}{1^{4/3}} = 2$$

Using Eq. (19), one may derive δ as:

$$\delta = \frac{k^2 h_c^{1/3}}{\alpha g} = \frac{73.3711103^2 \times 1.82756233^{1/3}}{2 \times 9.81} = 335.460903$$

According to Eq. (38), the following results are derived:

$$F\left(h_1^*, 0\right) = \frac{3}{4} h_1^{*\,4/3} - \frac{3}{19} h_1^{*\,19/3} - \frac{45}{76} = \frac{3}{4} - \frac{3}{19} - \frac{45}{76} = 0$$

$$F\left(h_2^*,0\right)=\frac{3}{4}h_2^{*\,4/3}-\frac{3}{19}h_2^{*\,19/3}-\frac{45}{76}$$
$$=\left|\frac{3}{4}\times 1.00547177-\frac{3}{19}\times 1.00547177-\frac{45}{76}\right|=7.54906\times 10^{-5}$$

Thus, one may write:

$$\left[F(h_2^*,0)-F(h_1^*,0)\right]=7.54906\times 10^{-5}$$

According to Eq. (36), the length L separating the depths h_1 and h_2 is as:

$$\overset{h_2}{\underset{h_1}{L}}=\underset{1-2}{\Delta L}=h_c\,\delta\left[F(h_2^*,0)-F(h_1^*,0)\right]$$

Whence:

$$\underset{1-2}{\Delta L}=1.82756233\times 335.460903\times 0.0000754906=0.04628147\,m$$

That is,

$$\overset{h_2}{\underset{h_1}{L}}=0+\underset{1-2}{\Delta L}=\underset{1-2}{\Delta L}=0.04628147\,m$$

Continue the calculation with the procedure described above, writing that:

$$\overset{h_3}{\underset{h_1}{L}}=\overset{h_2}{\underset{h_1}{L}}+\underset{2-3}{\Delta L}$$

More generally, one may write:

$$\overset{h_{i+1}}{\underset{h_1}{L}}=\overset{h_i}{\underset{h_1}{L}}+\underset{h_i\to h_{i+1}}{\Delta L}\ ,\ i=1,2,\ldots$$

Note that for $i = 1$, $\overset{h_1}{\underset{h_1}{L}}=0$.

Based on this calculation procedure, table 1 was dressed.

Table 1: H2-type backwater curve calculations according to the advocated method

$Q = 10\,m^3/s$, m = 1, ε = 2 mm, L = 5 m, $h_c = 1.82756233\,m$, $\Delta h = 0.01\,m$

h_1 (m)	h_2 (m)	h_1^*	h_2^*	δ	I	ΔL (m)	L (m)
1.82756233	1.82756233	1	1	335.460903	0	0	0
1.82756233	1.83756233	1	1.00547177	335.460903	7.5491E-05	0.04628147	0.04628147
1.83756233	1.84756233	1.00547177	1.01094354	335.460903	0.00022905	0.14042797	0.18670944
1.84756233	1.85756233	1.01094354	1.01641531	335.460903	0.00038655	0.23698191	0.42369135
1.85756233	1.86756233	1.01641531	1.02188708	335.460903	0.00054803	0.33598651	0.75967786
1.86756233	1.87756233	1.02188708	1.02735885	335.460903	0.00071359	0.43748555	1.19716341
1.87756233	1.88756233	1.02735885	1.03283062	335.460903	0.00088329	0.54152339	1.7386868
1.88756233	1.89756233	1.03283062	1.03830239	335.460903	0.0010572	0.6481449	2.3868317
1.89756233	1.90756233	1.03830239	1.04377416	335.460903	0.0012354	0.75739557	3.14422727
1.90756233	1.91756233	1.04377416	1.04924593	335.460903	0.00141797	0.86932143	4.0135487
1.91756233	1.92756233	1.04924593	1.0547177	335.460903	0.00160497	0.98396908	**4.99751778**

The curve representing $h = f(L)$ is plotted in Fig. 3 according to the values of Table 1. Over a length of 4.99751778 m ≈ 5 m, the surface water profile evolves from the critical depth $h_c = 1.82756233\,m$ in the control section to the final depth $h = 1.92756233\,m$. The evolution takes place according to the horizontal H2-type backwater curve.

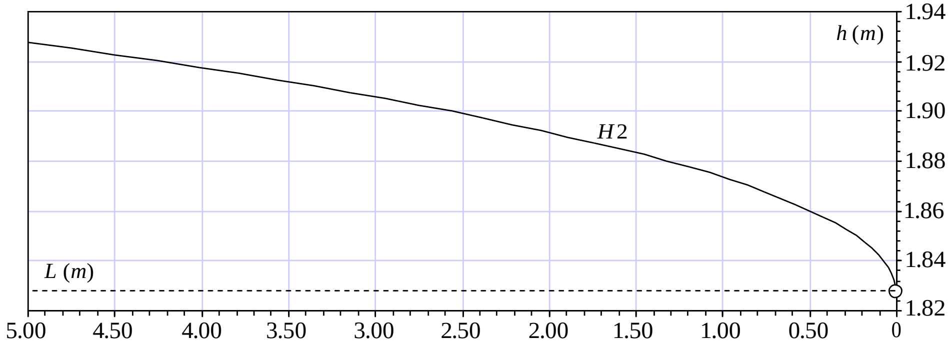


Figure 3: H2-type backwater curve according to the data of the considered example. (o) Control section $h_c = 1.82756233\,m$

Let us check the calculations by directly using Eq. (39):

$$\overset{h_2}{\underset{h_1}{L}} = h_c\delta\left[\frac{3}{4}\left(h_2^{*4/3} - h_1^{*\,4/3}\right) - \frac{3}{19}\left(h_2^{*19/3} - h_1^{*19/3}\right)\right] \tag{39}$$

where h_2^* = 1.0547177; h_1^* = 1; δ= 335.460903; and h_c = 1.82756233 m

In absolute value, the calculation gives: L = 4.99751857 m. This is indeed the value almost equal to that reported in Table 1, i.e., L = 4.99751778 m.

Another check is to directly use Eq. (34):

$$\int_{h_1^*}^{h_2^*} dL^* = \delta \int_{h_1^*}^{h_2^*} \left(h^{*1/3} - h^{*16/3}\right) dh^* \tag{34}$$

Thus:

$$L = h_c\delta \int_{h_1^*}^{h_2^*} \left(h^{*1/3} - h^{*16/3}\right) dh^* \tag{55}$$

With h_1^*=1 and h_2^*= 1.0547177 corresponding to the final relative depth (Table 1), the value of the previous integral calculated by the appropriate software is such that:

$$\int_{h_1^*}^{h_2^*} \left(h^{*1/3} - h^{*16/3}\right) dh^* = -0.00815155$$

Whence:

$$L = h_c\delta \int_{h_1^*}^{h_2^*} \left(h^{*1/3} - h^{*16/3}\right) dh^* = 1.82756233 \times 335.460903 \times 0.00815155 = 4.9975173\, m$$

This value is almost identical to that calculated by the previous procedure and reported in table 1, i.e., L = 4.99751778 m.

EXAMPLE 2

This example concerns the calculation of the C-type backwater curve by the advocated method. The calculation procedure will be identical to that described in example 1. The example of the C1-type backwater curve will be taken and its first depth will be defined as h_1 = 1.80 m. The following depths will be calculated considering a depth step $\Delta h = 0.01 m$. Therefore, for depths, one may write:

$$h_{i+1} = h_i + \Delta h \text{ , } i = 1,2,...$$

On the other hand, for lengths, on may write:

$$L_{h_1}^{h_{i+1}} = L_{h_1}^{h_i} + \Delta L_{h_i \to h_{i+1}} \text{ , } i = 1, 2,\dots$$

Consider a triangular channel with slope S_0 = 0.0035 and an apex angle θ = 90° corresponding to m = 1. The value of α is therefore the same as that calculated in example 1, i.e. α = 2. The absolute roughness ε was estimated to be ε = 2 mm corresponding to a Strickler's coefficient $k = 73.3711103 m^{1/3}/s$ according to Eq. (54). The flow rate flowing through the channel is Q = 3 m^3/s.

Calculate the length L separating the depths h_1 = 1.80 m and h_2 = 1.81 m.

Perform the calculation using the differential equation that governs the gradually varying flow as well as the approximate relationships proposed in the theoretical part of the study.

Solution

According to Eq. (12), the critical flow depth is as follows:

$$h_c = \left(\frac{2Q^2}{g\,m^2} \right)^{1/5} = \left(\frac{2 \times 3^2}{9.81 \times 1^2} \right)^{1/5} = 1.12906956\, m$$

According to Eq. (19), the dimensionless parameter δ is as follows:

$$\delta = \frac{k^2 h_c^{1/3}}{\alpha g} = \frac{73.3711103^2 \times 1.12906956^{1/3}}{2 \times 9.81} = 285.709543$$

According to Eq. (32), one may derive the following:

$$\beta = S_0\,\delta = 0.0035 \times 285.709543 = 1$$

Since β = 1, it is therefore concluded that the slope of the channel is critical, meaning that:

$$S_0 = S_c = S \text{ , } \beta = S_0 / S_c = 1 \text{ , } h_n^* = 1 \text{, i.e. } h_n = h_c$$

Moreover:

$$h_1^* = h_1 / h_c = 1.80 / 1.12906956 = 1.59423304$$

Thus, in this study section, attention is given to the following case:

$$h > h_n = h_c \text{ ; i.e. } h^* > 1$$

Therefore, equation (44) applies:

$$dL^* = \frac{1}{S}\frac{h^{*16/3} - h^{*1/3}}{h^{*16/3} - 1} dh^* \tag{44}$$

That is:

$$L = \frac{h_c}{S}\int_{h_1^*}^{h_2^*} \frac{h^{*16/3} - h^{*1/3}}{h^{*16/3} - 1} dh^* \tag{56}$$

Let us recall Eq. (40):

$$F(h^*, 1) = \int_{h_1^*}^{h_2^*} \frac{h^{*16/3} - h^{*1/3}}{h^{*16/3} - 1} dh^* \tag{40}$$

Taking into account Eqs. (40) and (56) yields:

$$L = \frac{h_c}{S} I \tag{57}$$

where:

$$I = \int_{h_1^*}^{h_2^*} \frac{h^{*16/3} - h^{*1/3}}{h^{*16/3} - 1} dh^* = \left[F(h_2^*, 1) - F(h_1^*, 1)\right] \tag{58}$$

In contrast:

$$h_2^* = h_2 / h_c = 1.81 / 1.12906956 = 1.60308989$$

With $h_1^* = 1.59423304$ and $h_2^* = 1.60308989$, using the appropriate software package, the integral I expressed by Eq. (58) is such that: I = 0.0087231

According to Eq. (57), the exact length L separating the two defined depths is then:

$$L_{h1}^{h2}(\text{exact}) = \frac{h_c}{S} I = \frac{1.12906956}{0.0035} \times 0.0087231 = 2.813996198\,m \approx 2.814\,m$$

For the sake of verification and confirmation of the result, let us use the approximate relationship (52):

$$L_{h_1}^{h_2} = (h_2^* - h_1^*)\left[0.022(h_2^* + h_1^*) + 0.9129\right]\frac{h_c}{S} \tag{52}$$

With $h_1^* = 1.59423304$ and $h_2^* = 1.60308989$, Eq. (52) gives:

$$L_{h_1}^{h_2} = (1.60308989 - 1.59423304) \times \left[0.022 \times (1.60308989 + 1.59423304) + 0.9129\right] \times$$

$$\frac{1.12906956}{0.0035} = 2.8092603\,m \approx 2.81\,m$$

Therefore, the use of the approximate relation (51) for calculating L causes a relative error of:

$$\frac{\Delta L}{L} = 100 \times \left|\frac{2.8092603 - 2.813996198}{2.813996198}\right| \approx 0.17\ \%$$

This clearly shows that the approximate relationship (52) is very reliable, at least for the example considered. Several numerical applications have shown this tendency. Moreover, if one uses the formal approximation (Eq. 53), only a relative error of 0.07 % in this case is made.

It should be noted that, for the considered C1-type backwater curve considered in the previous example, the depth increases by only 1 cm over a distance of approximately L = 2.8 m.

CONCLUSION

The objective of the article was to derive a generalized theoretical approach to the calculation of backwater curves evolving in a triangular channel. The approach is based on the introduction of dimensionless parameters to give it a character of general validity. Thanks to the parameter h^* which represents the ratio between depth h and critical depth h_c, the differential equation governing the phenomenon has been transformed into an equation of dimensionless terms. The energy slope S_f was expressed by Strickler's equation, which is valid in the rough turbulent domain. The final equation [Eq. (26)], allowing calculating the length L separating two given depths, presents only two dimensionless compound parameters namely δ and β whose determination is easy [Eqs.

(19) and (32)]. In addition, the dimensionless parameter β has been shown to be closely related to the normal depth [Eq. (32)] and above all enables easy slope classification.

The particular case of the horizontal triangular channel has been the subject of particular attention. The differential equation is simplified, and its resolution is made possible by the judicious choice of a constant. The graphical representation of the derived differential equation resulted in nondimensional gradually varying flow profiles representing H2-type backwater curves ($dh/dL > 0$) for h^* values greater than 1 ($h^* > 1$), as well as H3-type drawdown curves ($dh/dL < 0$) for h^* values less than 1 ($h^* < 1$) [Figs. 1a and 1b].

Similarly, the particular case of the channel with a critical slope has been the subject of intense investigation showing the characteristics of the backwater curve of the type C1-type backwater curve. It has been shown that for the singular point corresponding to $h^* = 1$, i.e., $h = h_c$, the limit of dh/dL is equal to $0.9375/S$, where S is the bed channel slope. To simplify the calculations, reliable approximate relationships have been proposed for the calculation of the length L separating two given depths [Eqs. (49), (50), (52), and (53)].

Numerical examples have been proposed at the end of the article to show how to calculate backwater curves using the advocated method.

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